Lecture 2
Variational Principles in Computational Solid Mechanics

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Lecture 2

Variational Principles in Computational Solid Mechanics

Chapters

1. **Principle of virtual work. Reciprocal Theorem.**

2. **The Least Action Principle in dynamics.**
   (Hamilton's principle and the Euler-Lagrange equation)

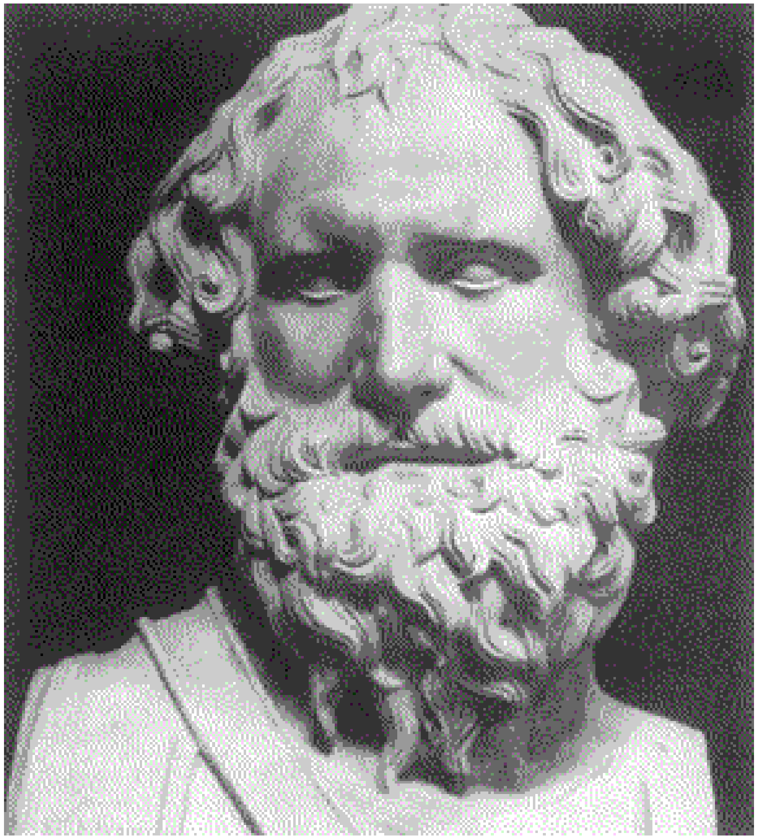
3. **Weighted Residual (Galerkin's) Method and the principle of minimum potential energy.**

4. **Element formulations using variational principles.**

Lecture 2

Chapter 1

Principle of virtual work.
Reciprocal Theorem.



Archimedes was the first person to use the concept of virtual work

1.1 Virtual work (over virtual displacements)

What is virtual displacement?

Virtual displacement is <u>an imaginary displacement</u> of any system <u>in frozen (fixed) time and space</u>, over and above the actual displacement.

In particular, virtual displacement can be imposed upon an actual displacement <u>at equilibrium</u>.

What is virtual work?

Virtual work is the work done by the forces (external and internal) traveling through virtual displacements.

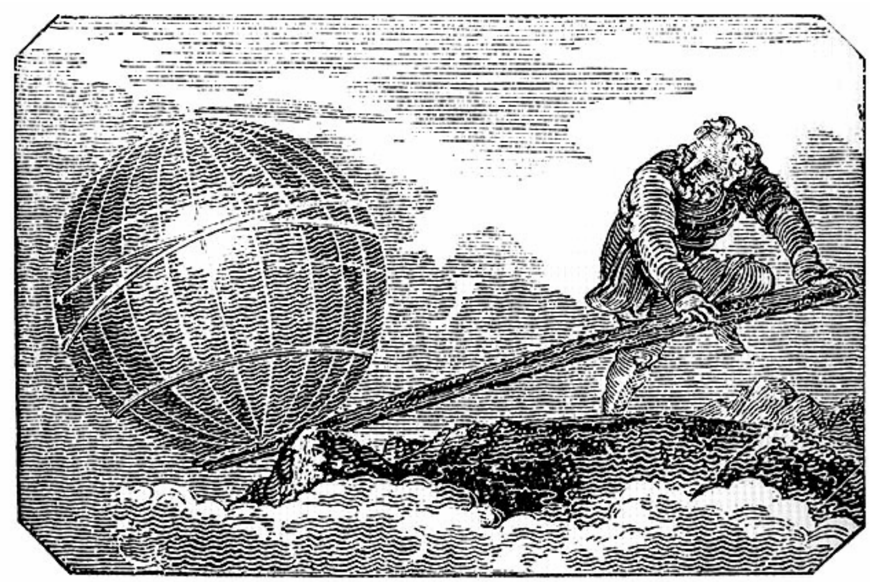
1.2 Principle of Virtual work

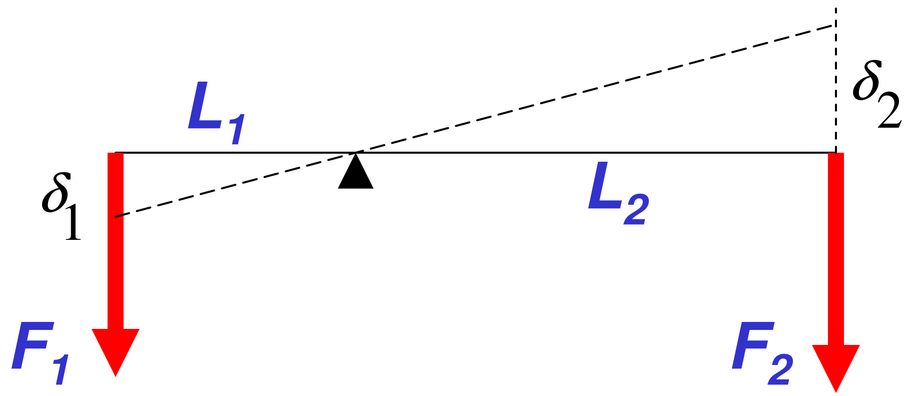
For any arbitrary yet admissible set of virtual displacements, superposed over the equilibrium configuration, <u>the net virtual work done by the forces (external and internal) is zero.</u>

Equilibrium ↔ *Zero* virtual work

$$\boxed{\delta W_{ext} - \delta W_{int} = 0} \qquad \textbf{(1.1)}$$

Archimedes was the first person to use the concept of Virtual Work in his calculations for the Lever.





Equilibrium: Virtual Work Principle

$$F_1\delta_1 - F_2\delta_2 = 0$$

"Give me a place to stand on, and I will move the Earth."

Geometric Compatibility

$$\frac{\delta_1}{L_1} = \frac{\delta_2}{L_2}$$

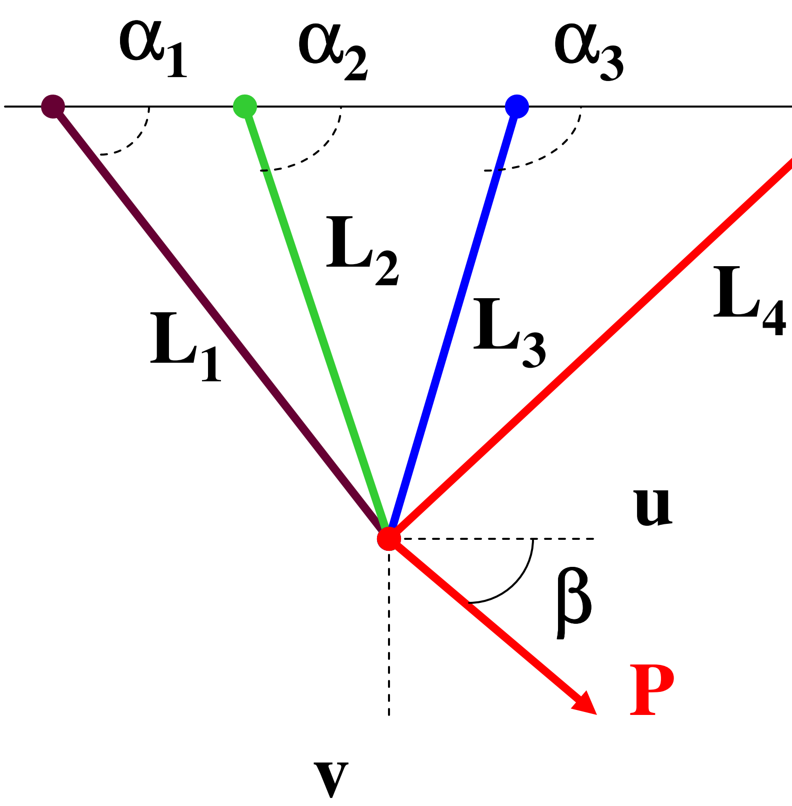
Together

$$\boxed{F_1 L_1 = F_2 L_2}$$

Example 1

Derive the equations of equilibrium for the bar system.

Suppose that actual orthogonal displacement components of load point at equilibrium are *u* and *v*.

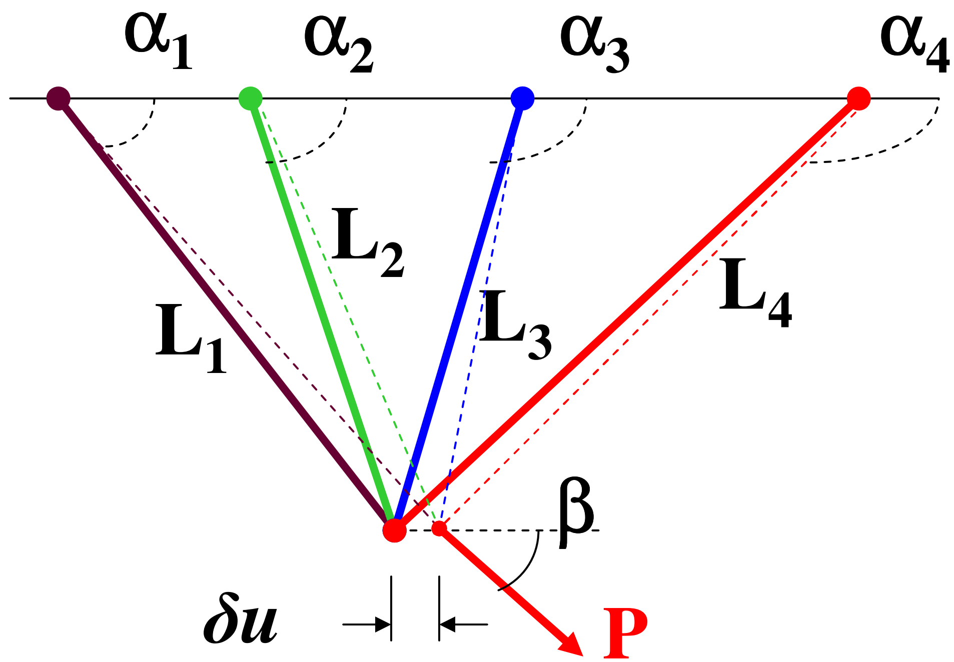


Actual strain (at equilibrium) in the *i*-th bar is

$$\varepsilon_i = \frac{u.\cos\alpha_i}{L_i} + \frac{v.\sin\alpha_i}{L_i}$$

Actual stress (at equilibrium) in the *i*-th bar is

$$\sigma_i = E_i\varepsilon_i = E_i\left(\frac{u.\cos\alpha_i}{L_i} + \frac{v.\sin\alpha_i}{L_i}\right)$$



Apply a virtual displacement δu along direction *u*.

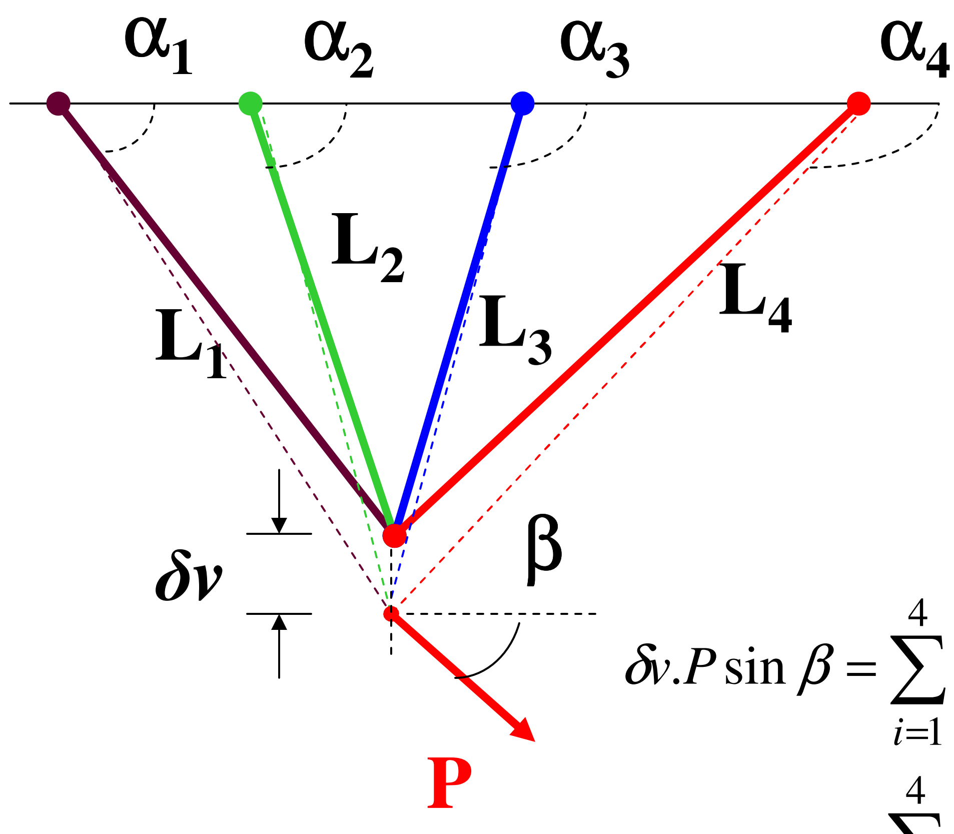
By *Principle of virtual work,*

$$\boxed{\delta W_{ext} = \delta W_{int}}$$

$$\delta u.P\cos\beta = \sum_{i=1}^{4} \sigma_i \delta\varepsilon_i.(A_i L_i)$$

$$= \sum_{i=1}^{4} E_i \left(\frac{u.\cos\alpha_i}{L_i} + \frac{v.\sin\alpha_i}{L_i} \right) \left(\frac{\delta u.\cos\alpha_i}{L_i} \right) (A_i L_i)$$

$$\boxed{P\cos\beta = \sum_{i=1}^{4} \frac{E_i A_i}{L_i} \left(u\cos^2\alpha_i + v\sin\alpha_i\cos\alpha_i \right)}$$



Apply a virtual displacement δv along direction *v*.

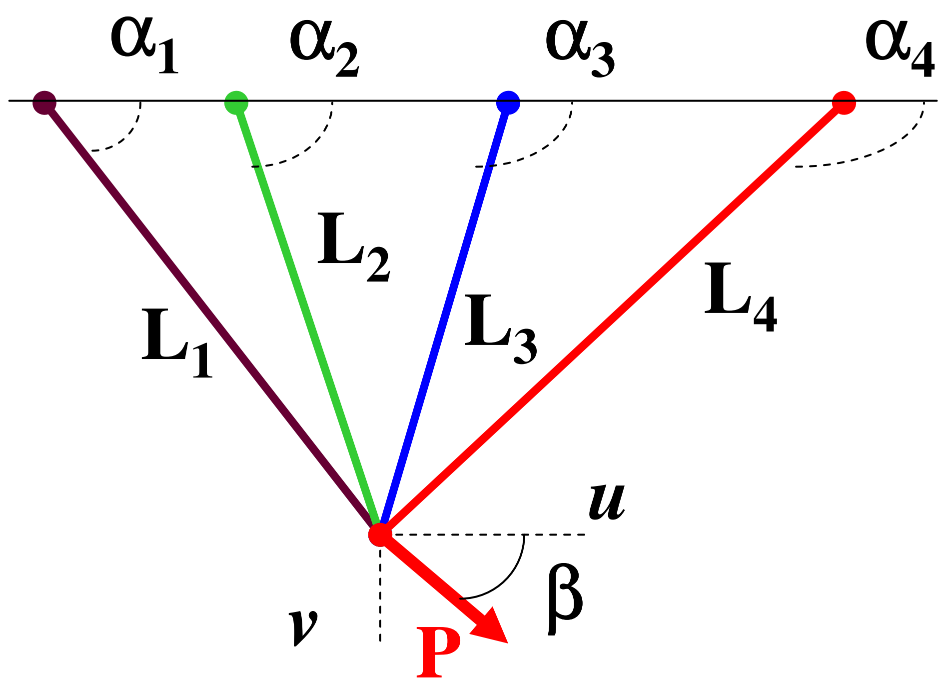
By *Principle of virtual work,*

$$\boxed{\delta W_{ext} = \delta W_{int}}$$

$$\delta v . P \sin\beta = \sum_{i=1}^{4} \sigma_i \delta\varepsilon_i .(A_i L_i)$$

$$= \sum_{i=1}^{4} E_i \left(\frac{u.\cos\alpha_i}{L_i} + \frac{v.\sin\alpha_i}{L_i} \right)\left(\frac{\delta v.\sin\alpha_i}{L_i} \right)(A_i L_i)$$

$$\boxed{P\sin\beta = \sum_{i=1}^{4} \frac{E_i A_i}{L_i}\left(u\sin\alpha_i \cos\alpha_i + v\sin^2\alpha_i \right)}$$



The equation of equilibrium for the bar system is

$$\begin{bmatrix} \sum_{i=1}^{4} \frac{E_i A_i}{L_i}\left(\cos^2 \alpha_i\right) & \sum_{i=1}^{4} \frac{E_i A_i}{L_i}\left(\sin \alpha_i . \cos \alpha_i\right) \\ \sum_{i=1}^{4} \frac{E_i A_i}{L_i}\left(\sin \alpha_i . \cos \alpha_i\right) & \sum_{i=1}^{4} \frac{E_i A_i}{L_i}\left(\sin^2 \alpha_i\right) \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} P \cos \beta \\ P \sin \beta \end{Bmatrix}$$

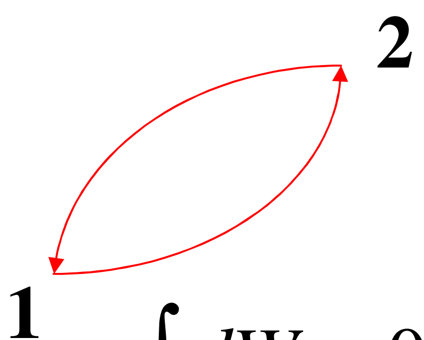
Solving this equation gives displacements u and v, from which the stresses in the bars can be determined.

1.3 Conservative Systems

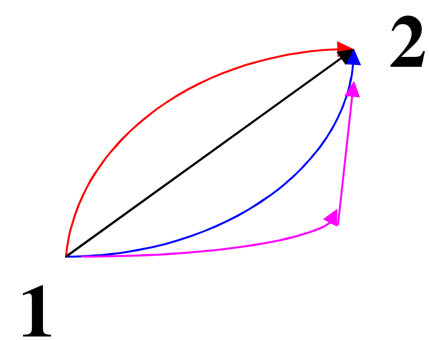
In a conservative system, the net work done by the forces in any closed path is zero.

In a conservative system, the net work from one equilibrium state to another is path independent.

$$\oint dW = 0 \quad \Rightarrow \quad \int_1^2 dW = -\int_2^1 dW \tag{1.2}$$



$$\oint dW = 0$$



$\int_1^2 dW$ **is path independent**

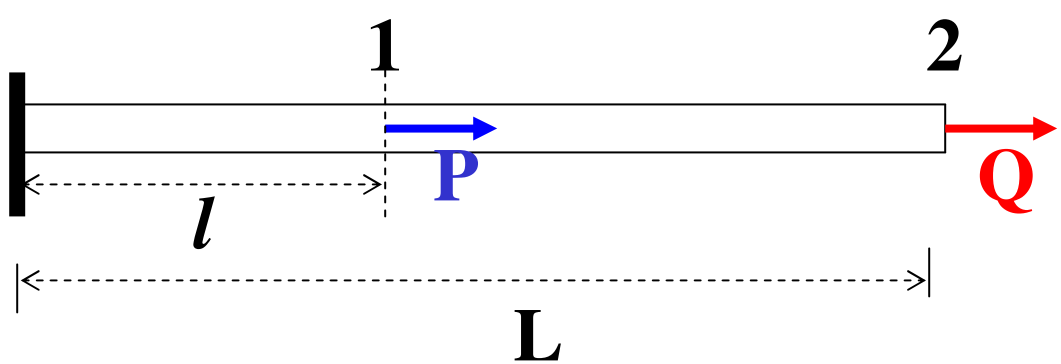
1.4 Symmetry in Linear Elastic Conservative Systems

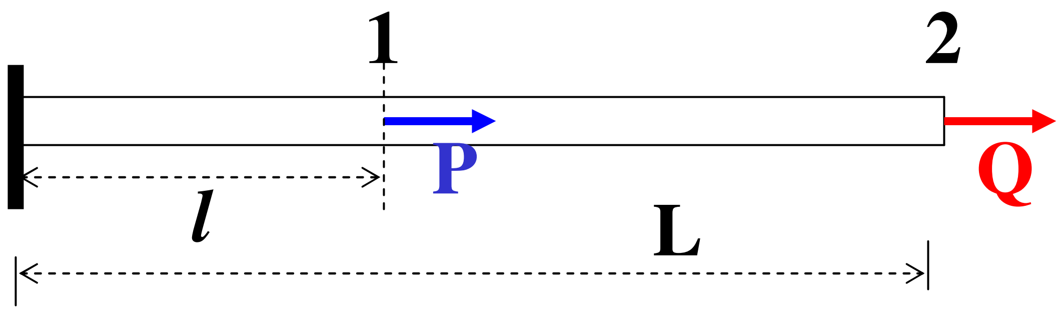
In a linear elastic conservative system, the stiffness and flexibility matrices are symmetric.

$$k_{ij} = k_{ji} \qquad\qquad f_{ij} = f_{ji} \tag{1.3}$$

Example 2

Demonstrate the invariance of work done in a bar with changing order of loading.





Case 1. **First load point 1 by P and then load 2 by Q.**
Work done by loads:

$$W_{PQ} = \frac{1}{2}P.u_{1,1} + P.u_{1,2} + \frac{1}{2}Q.u_{2,2} = \frac{1}{2}P\left(\frac{Pl}{EA}\right) + P\left(\frac{Ql}{EA}\right) + \frac{1}{2}Q\left(\frac{QL}{EA}\right)$$

Case 2. **First load point 2 by Q and then load 1 by P.**
Work done by loads:

$$W_{QP} = \frac{1}{2}Q.u_{2,2} + Q.u_{2,1} + \frac{1}{2}P.u_{1,1} = \frac{1}{2}Q\left(\frac{QL}{EA}\right) + Q\left(\frac{Pl}{EA}\right) + \frac{1}{2}P\left(\frac{Pl}{EA}\right)$$

$$\boxed{W_{PQ} = W_{QP}}$$

This demonstrates the invariance of work done with changing order of loading.

Symmetry in flexibility

$$f_{12} = \frac{u_{1,2}}{Q} = \left(\frac{l}{EA}\right)$$

$$f_{21} = \frac{u_{2,1}}{P} = \left(\frac{l}{EA}\right)$$

$$\longrightarrow \quad f_{12} = f_{21}$$

1.5 Maxwell Betti's Reciprocal Theorem

On a given structure (linear and conservative system), the virtual work done by load system A over virtual displacement for load system B is equal to the virtual work done by load system B over virtual displacement for load system A.

$$\boxed{\{\delta_B\}^T\{F_A\} = \{\delta_A\}^T\{F_B\}} \qquad \textbf{(1.4)}$$

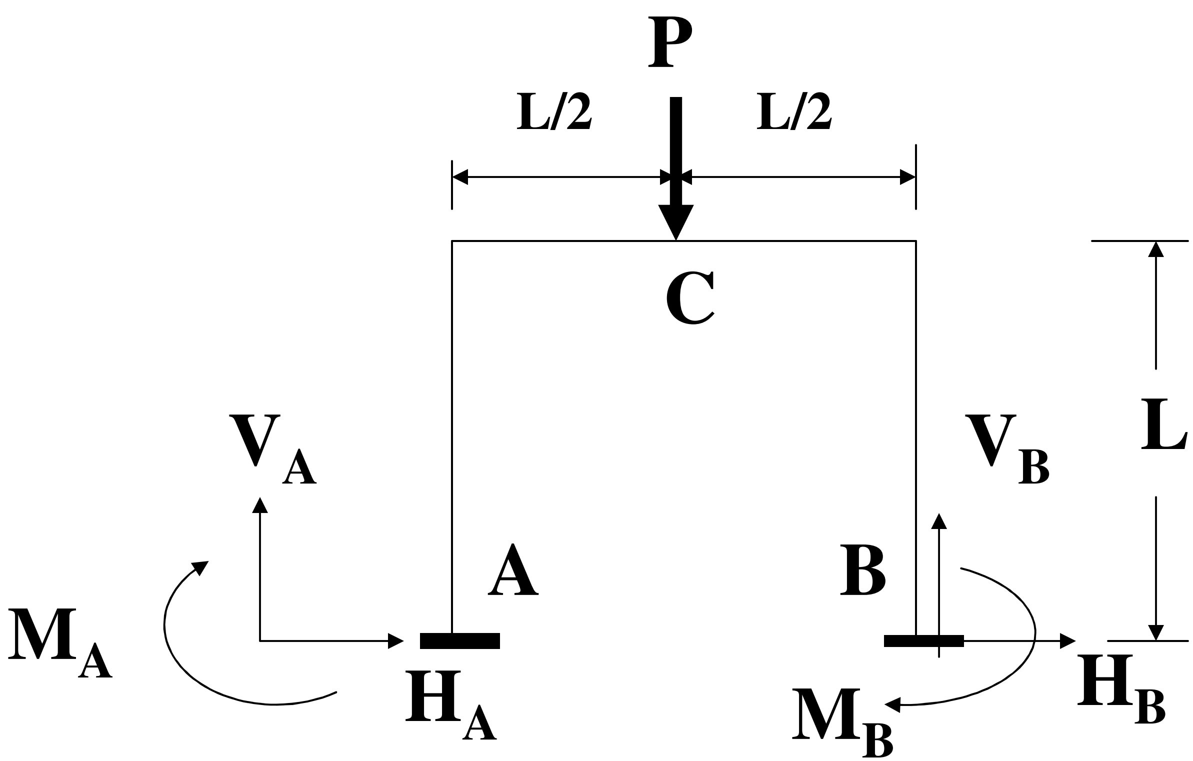
Proof

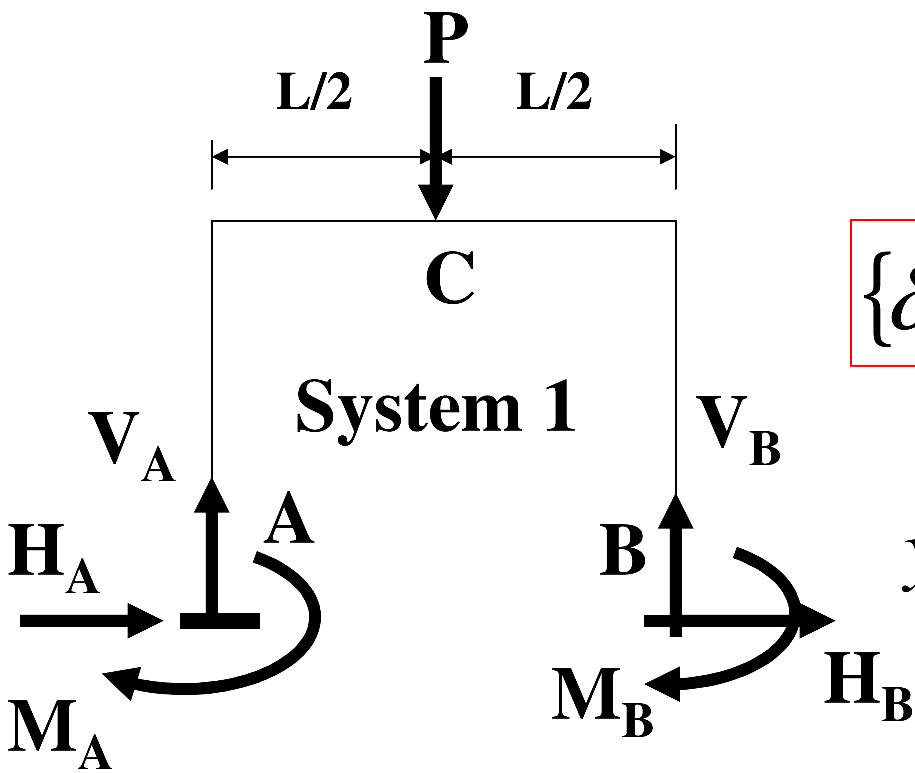
$$\{\delta_B\}^T\{F_A\} = \{\delta_B\}^T[K]\{\delta_A\} = \{\delta_A\}^T[K]\{\delta_B\}$$
$$= \{\delta_A\}^T\{F_B\}$$

…because stiffness [K] is symmetric (linear conservative system)

Example 3

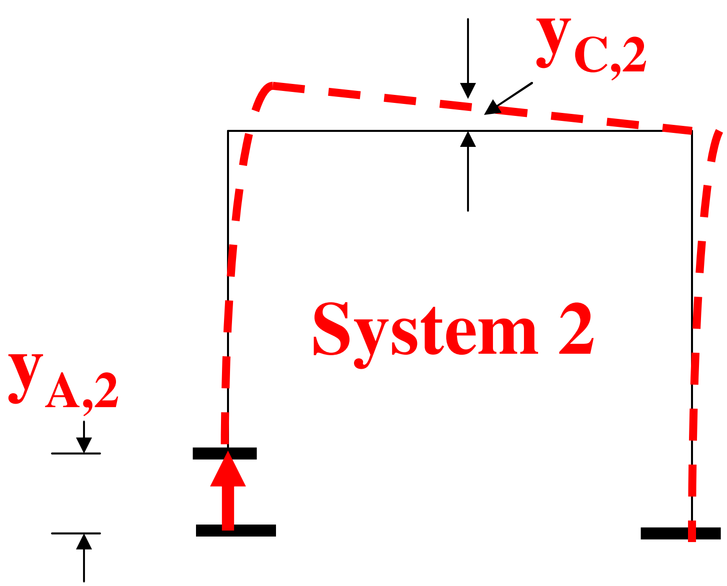
Determine the support reactions of the portal frame under the loading shown.



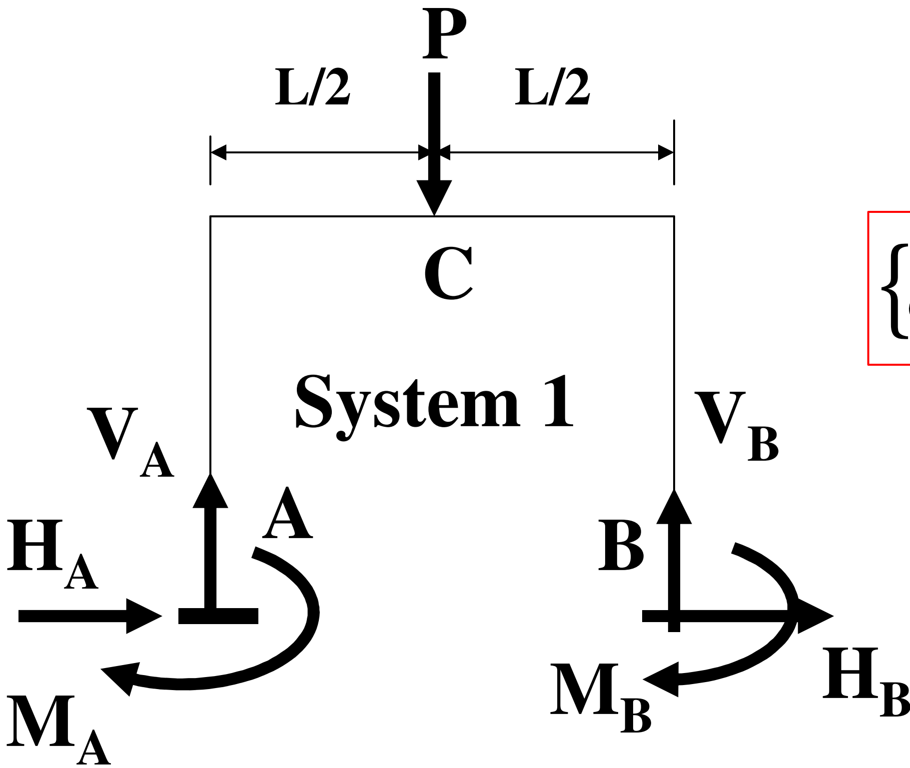


$$\{\delta_2\}^T\{F_1\} = \{\delta_1\}^T\{F_2\}$$

$$y_{A,2}.V_A - y_{C,2}.P = 0$$

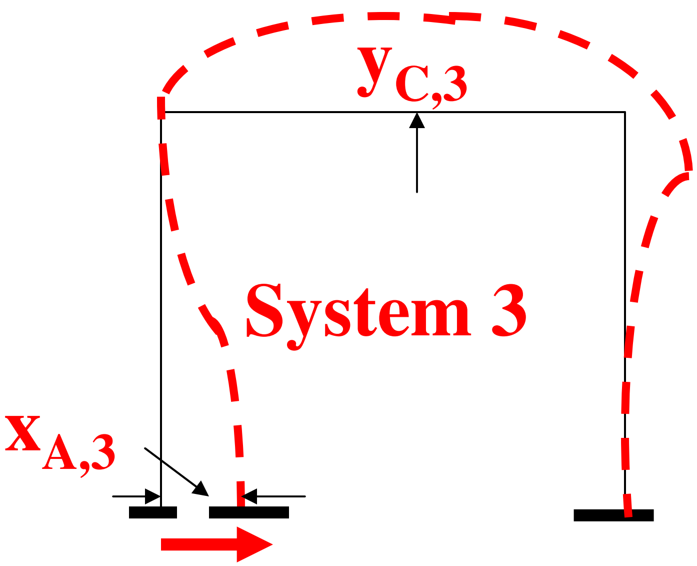


$$V_A = \frac{P.y_{C,2}}{y_{A,2}}$$

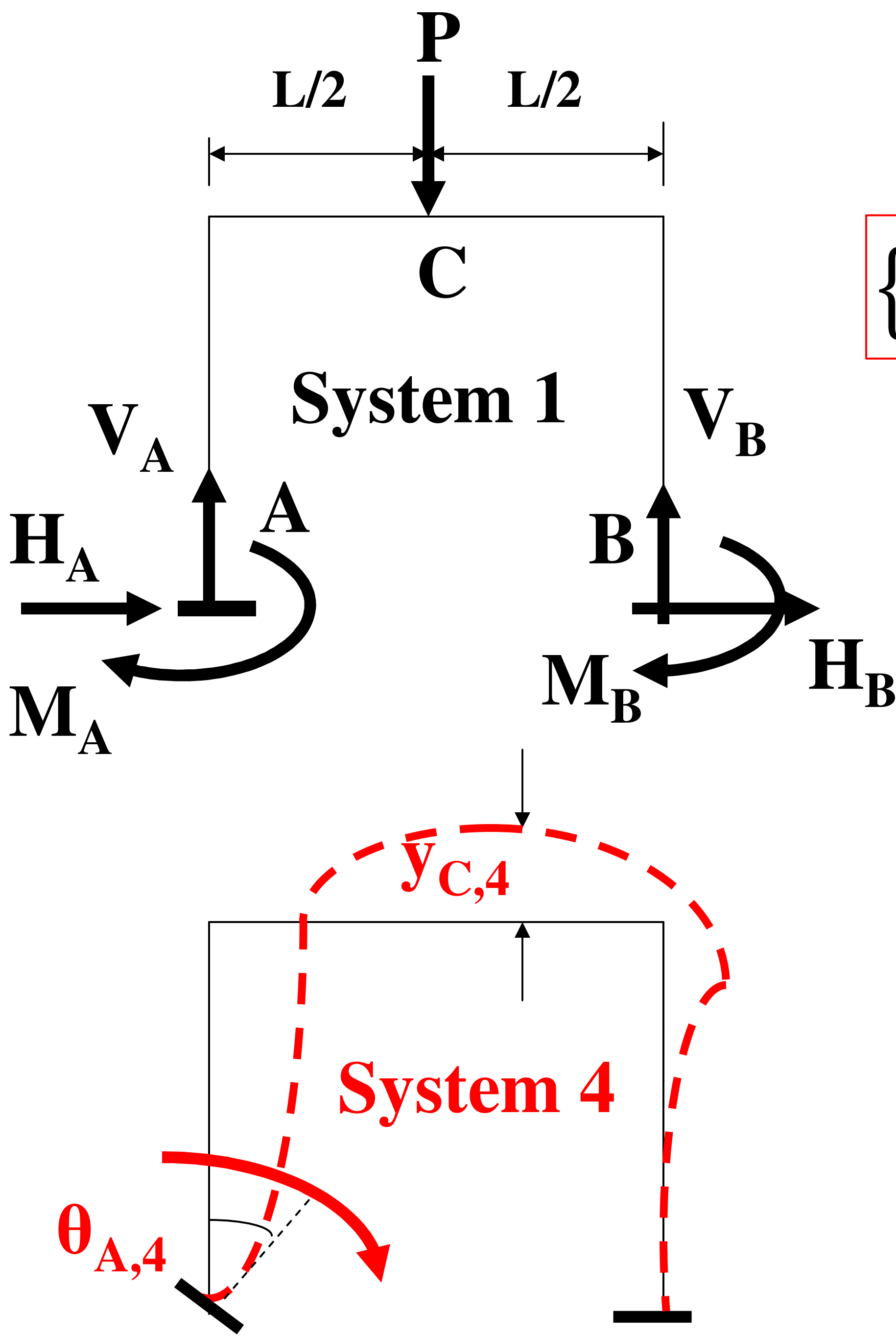


$$\boxed{\{\delta_3\}^T\{F_1\}=\{\delta_1\}^T\{F_3\}}$$

$$x_{A,3}.H_A - y_{C,3}.P = 0$$



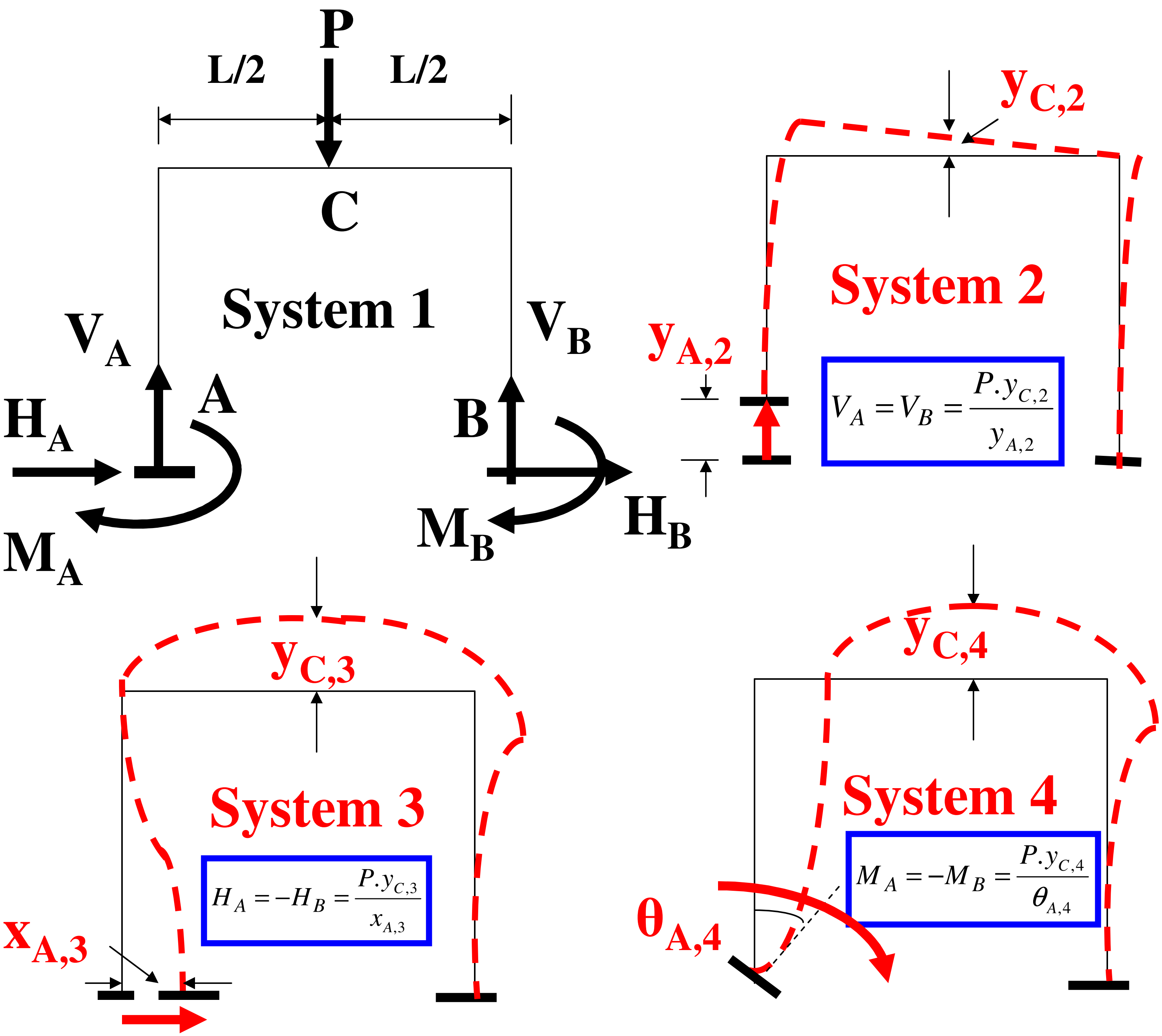
$$\boxed{H_A = \frac{P.y_{C,3}}{x_{A,3}}}$$



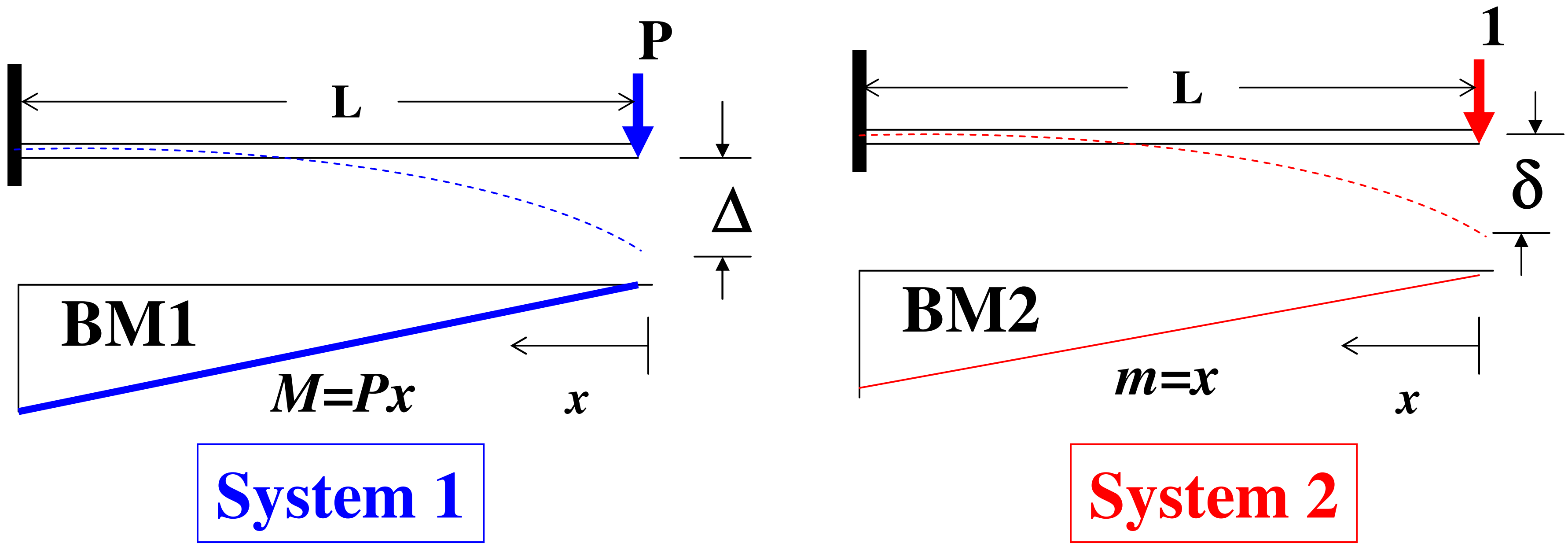
$$\boxed{\{\delta_4\}^T\{F_1\}=\{\delta_1\}^T\{F_4\}}$$

$$\theta_{A,4}.M_A - y_{C,4}.P = 0$$

$$\boxed{M_A = \frac{P.y_{C,4}}{\theta_{A,4}}}$$


P
L/2
L/2
C
System 1
V_A
H_A
A
M_A
V_B
B
M_B
H_B
$y_{C,2}$
System 2
$y_{A,2}$
$V_A = V_B = \frac{P.y_{C,2}}{y_{A,2}}$
$y_{C,3}$
System 3
$H_A = -H_B = \frac{P.y_{C,3}}{x_{A,3}}$
$x_{A,3}$
$y_{C,4}$
System 4
$M_A = -M_B = \frac{P.y_{C,4}}{\theta_{A,4}}$
$\theta_{A,4}$

1.6 Consequence of the Reciprocal Theorem - The unit load method



By Reciprocal Theorem: $\Delta.1 = \delta.P$

But $\delta.P = (\text{slope2}).(\text{BM1})$ $= \int_0^L \left(\frac{m(x)}{EI} dx \right) M(x)$

$$\boxed{\Delta = \int_0^L \left(\frac{M(x).m(x)}{EI} dx \right) = \int_0^L \left(\frac{Px.x}{EI} dx \right) = \frac{PL^3}{3EI}} \qquad \textbf{(1.5)}$$

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Lecture 2

Chapter 2

The Least Action Principle in dynamics

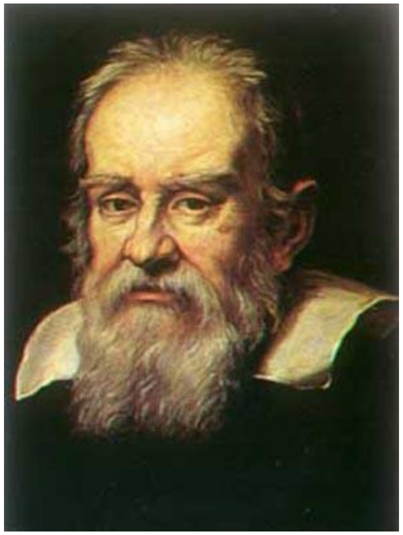
(Hamilton's principle and the Euler-Lagrange equation)

"Nature always acts by the shortest paths".

. - Pierre de Fermat

The pioneers of Analytical Mechanics & Variational Calculus

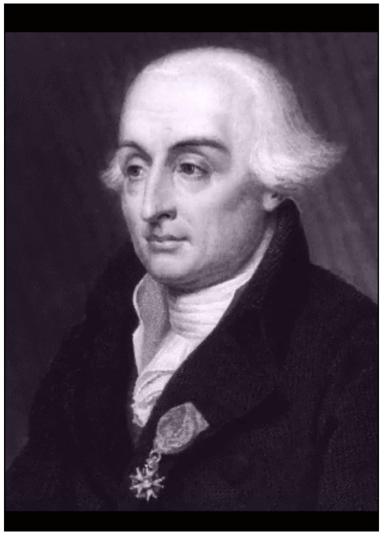
Maupertuis



Galileo

Newton

Fermat



Lagrange

Euler

Hamilton

2.1 The Least Action Principle

Early History of Variational Calculus

Archimedes (287-212 BC)

For the first time in history the principle of virtual work (from virtual displacements) was discovered and employed by Archimedes to derive the Lever Rule.

$$F_1 L_1 = F_2 L_2 \qquad \left[F_1 \delta_1 + F_2(-\delta_2) = 0 \qquad \delta_1 / L_1 = \delta_2 / L_2\right]$$

Pierre de Fermat (1601-1665 AD)

Discovered the Principle of Least time for optical path. Derived Snell's Law of Refraction (and reflection) of light rays using this principle.

"*Nature always acts by the shortest paths*".

$$\delta \int_A^B \frac{dl}{v} = 0 \qquad \mu_1 \sin\theta_1 = \mu_2 \sin\theta_2$$

Pierre Louis Moreau de Maupertuis (1698-1759AD)

Defined a parameter called the Action: (Action=Path Integral of the momentum)

He postulated that Nature always behaves in a way so as to optimize (minimize) this action.

(Euler presented the mathematical proof of this postulate.)

$$I = \int_A^B mv.dl \quad ; \quad \delta \int_A^B mv.dl = 0$$

Johann Bernoulli (1667-1748)

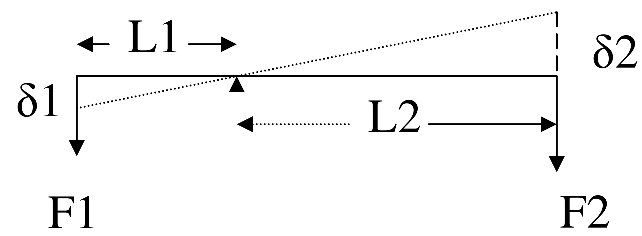
Presented the Brachistochrone problem (finding the path of fastest descent between two fixed points) in *Acta Eruditorum* and its solution is 1682. Solutions were provided independently by

Isaac Newton, Jacob Bernoulli (elder brother of Johann) and L' Hospital.

Archimedes' Lever

Equilibrium of any system under given forces conforms to the vanishing of the Net Virtual work done by these forces traveling over the geometrically compatible virtual displacements:

$$F_1 L_1 = F_2 L_2 \qquad [F_1\delta_1 + F_2(-\delta_2) = 0 \qquad \delta_1 / L_1 = \delta_2 / L_2]$$



Fermat's Optical Path

From one fixed point to another, Light travels along the path of least time joining these points.

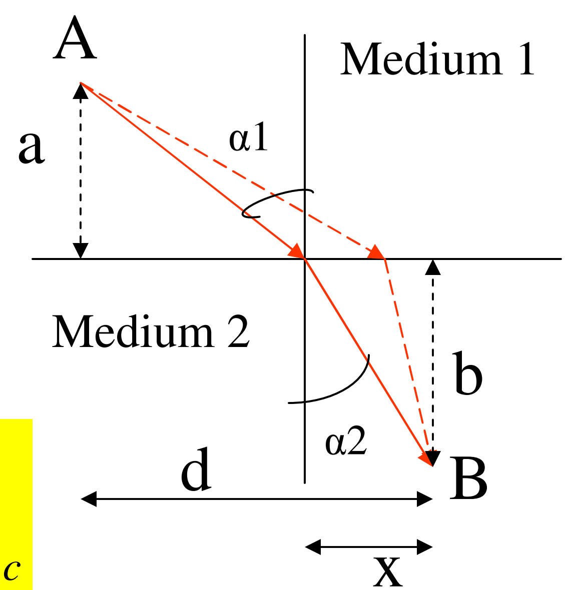
$$T_{AB} = \frac{Length}{Speed} = \frac{\sqrt{a^2 + (d-x)^2}}{v_1} + \frac{\sqrt{b^2 + x^2}}{v_2}$$

$$\frac{dT_{AB}}{dx} = \frac{(x-d)}{v_1\sqrt{a^2 + (d-x)^2}} + \frac{x}{v_2\sqrt{b^2 + x^2}}$$

For the optimized path of least time

$$\frac{dT_{AB}}{dx} = 0 \Rightarrow \qquad \frac{(d-x)}{v_1\sqrt{a^2 + (d-x)^2}} = \frac{x}{v_2\sqrt{b^2 + x^2}}$$

$$\frac{\sin\alpha_1}{v_1} = \frac{\sin\alpha_2}{v_2}$$



c =speed of light in vacuum
v =speed of light in medium

$$c\frac{\sin\alpha_1}{v_1} = c\frac{\sin\alpha_2}{v_2}$$

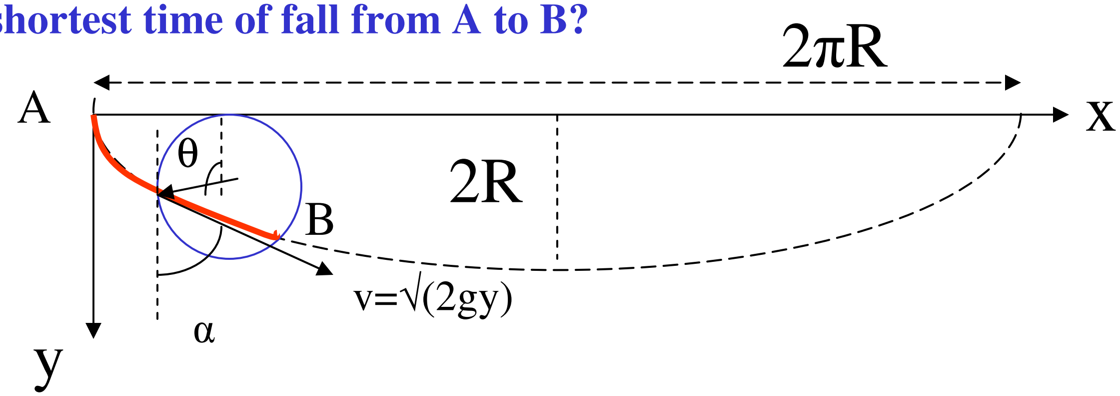
$$\mu_1 \sin\alpha_1 = \mu_2 \sin\alpha_2 \qquad \mu = \frac{c}{v}$$

$$\mu \sin\alpha = Consant$$

The Brachistochrone

The Path of Fastest Descent under gravity from one point to another.

- **Given two fixed points A and B.**
- **What is the path for shortest time of fall from A to B?**



For the path for shortest time of fall from A to B

$$\delta\int_A^B \frac{dl}{v} = 0$$

$$\Rightarrow \frac{\sin\alpha}{v} = k(Consant)$$

$$Speed: \quad v = \sqrt{2gy}$$

x=R(θ-sin θ)
y=R(1-cosθ)

The Brachistochrone is a Cycloid between points A and B.

Radius of generating wheel R=0.25/(k*k*g)

Hamilton's Principle and the Euler-Lagrange Equations

Hamilton's Principle of Least Action:

Nature determines the path of any particle from point 1 to point 2 in a way so as to minimize the Action I

L=Lagrangian, T=Kinetic Energy, V=Potential Energy

$$I = \int_1^2 L(q,\dot{q},t)dt \qquad L = T - V \qquad q = q(t)$$

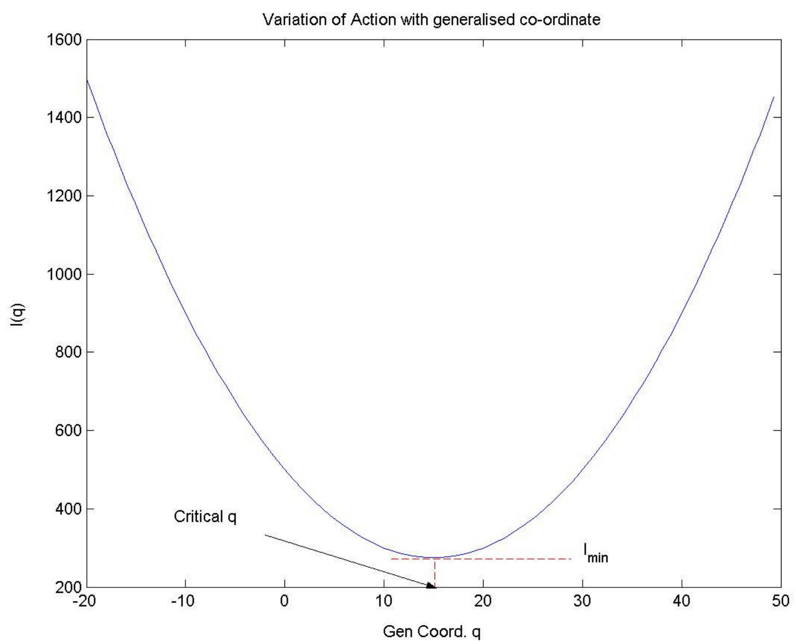
$$\Delta I \quad (for \quad virtual \quad changes \quad about \quad any \quad path) = \delta I + \delta^2 I + ..$$

For the Action to be a minimum, any virtual changes about the minimum point should lead to a net increase in the Action,

i.e.

$$\Delta I \quad (for \quad virtual \quad changes \quad about \quad \min) \geq 0.$$

$$\Delta I > 0 \qquad \delta I = 0 \qquad \delta^2 I > 0$$



i.e. The first variation of the Action (δI) should vanish for small perturbations (virtual displacements) about the minimum point.

First variation for the Lagrangian L (for virtual displacement in frozen time is)

$$\boxed{\delta L = \sum_{i=1}^{n}\left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}}\delta \dot{q}_i\right)}$$

subject to $\delta q_i(t=t_1) = \delta q_i(t=t_2) = 0$

Action $I = \int_1^2 L(q_i, \dot{q}_i, t)\,dt \qquad L = T - V \qquad q_i = q_i(t) \quad : generalised \quad co-ordinates$

First variation for the Action for any path 1-2 is

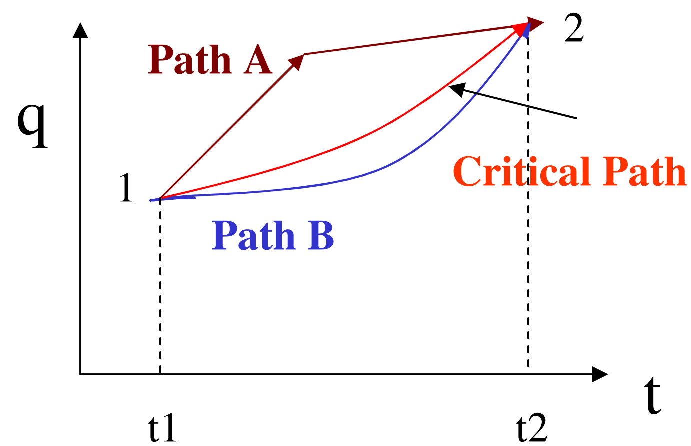
$$\delta I = \int_1^2 \delta L(q_i, \dot{q}_i, t)dt = \sum_{i=1}^{n}\int_1^2\left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}}\delta \dot{q}_i\right)dt$$

For the critical path, δI=0

$$0 = \sum_{i=1}^{n}\int_1^2\left(\frac{\partial L}{\partial q_i}\delta q_i\right)dt + \sum_{i=1}^{n}\left\{\left[\delta q_i \frac{\partial L}{\partial \dot{q}}\right]_1^2 - \int_1^2 \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\delta q_i dt\right\}$$

$$0 = \sum_{i=1}^{n}\int_1^2\left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i dt + \sum_{i=1}^{n}\left[\delta q_i \frac{\partial L}{\partial \dot{q}_i}\right]_1^2$$

$$\left[\delta q_i \frac{\partial L}{\partial \dot{q}_i}\right]_1^2 = 0 \qquad \because \delta q_i(t=t_1) = \delta q_i(t=t_2) = 0$$



Euler-Lagrange Equation (Conservative Systems)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

Euler-Lagrange Equation (Non-Conservative Systems)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i$$

$$Q_i = Non-conservative \quad generalised \quad force$$

What is a generalized force?

Virtual work=(Virtual physical displacement) $\times$ **Physical force**

=(Virtual generalized displacement) $\times$ **Generalized force**

$$= (\delta q_i).Q_i$$

2.2 Applications of the Least Action Principle

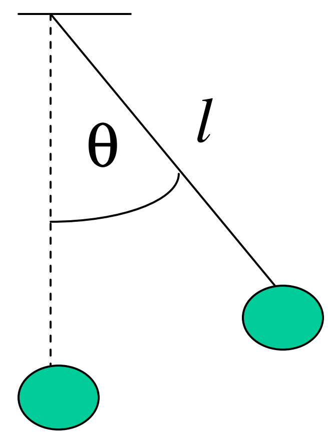
Free Vibration of Single Degree of freedom systems

The Pendulum

Kinetic Energy: $T = \frac{1}{2} m \dot{x}^2 = \frac{1}{2} m \left(l \dot{\theta} \right)^2$ **Potential Energy:** $V = mgl(1 - \cos\theta)$

Euler-Lagrange Equation (Conservative Systems) $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} == 0$ $L = T - V$

Equation of Motion $ml^2 \ddot{\theta} + mgl.\sin\theta = 0 \quad \Rightarrow \quad \ddot{\theta} + \frac{g}{l}.\sin\theta = 0$



Linear Analysis (small oscillations)

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \qquad \theta = A.\sin(\omega_n t + \varepsilon)$$

Natural frequency and Time Period

$$\omega_n = \sqrt{\frac{g}{l}}(rad/\sec) \qquad f_n = \frac{1}{2\pi}\sqrt{\frac{g}{l}}(Hz) \qquad \tau = \frac{1}{f_n} = 2\pi\sqrt{\frac{l}{g}}(\sec)$$

2.3 Some simple applications of the Least Action Principle

Free Vibration of Single Degree of freedom systems

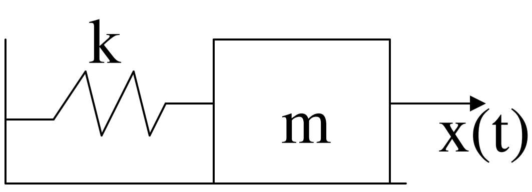
Example 1   The Spring-Mass model

Kinetic Energy: $T = \frac{1}{2} m\dot{x}^2$

Potential Energy: $V = \frac{1}{2} kx^2$

Euler-Lagrange Equation (Conservative Systems) $\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$ $\quad L = T - V$

Equation of Motion (free vibration) $m\ddot{x} + kx = 0$



Solution (free vibration) $x = A.\sin(\omega_n t + \varepsilon)$

Natural frequency and Time Period

$$\omega_n = \sqrt{\frac{k}{m}}\,(rad/\sec) \qquad f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}\,(Hz) \qquad \tau = \frac{1}{f_n} = 2\pi\sqrt{\frac{m}{k}}\,(\sec)$$

Free Vibration of Multi-degree of freedom systems

Example 2 The Spring-Mass model for a 2 DOF system

Kinetic Energy: $T = \frac{1}{2} m_1 \dot{x}_1^{\,2} + \frac{1}{2} m_2 \dot{x}_2^{\,2}$

Potential Energy: $V = \frac{1}{2} k_1 x_1^{\,2} + \frac{1}{2} k_2 (x_2 - x_1)^2$

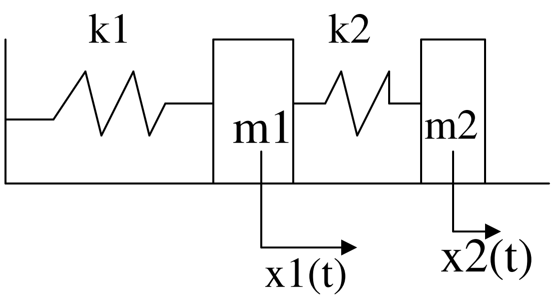
Euler-Lagrange Equation (Conservative Systems)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} - \frac{\partial L}{\partial x_1} = 0 \qquad L = T - V$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} - \frac{\partial L}{\partial x_2} = 0$$

Equation of Motion (free vibration)

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$



Solution

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} . \sin(\omega_n t + \varepsilon)$$

$$-\omega_n^{\,2}\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \Rightarrow \det\left\| \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} - \omega_n^{\,2}\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \right\| = 0$$

Eigenvalues and Eigen-modes

$$\omega_1^{\,2}: \quad \{\phi_1\} = \begin{Bmatrix} A_{1,1} \\ A_{2,1} \end{Bmatrix} \qquad \omega_2^{\,2}: \quad \{\phi_2\} = \begin{Bmatrix} A_{1,2} \\ A_{2,2} \end{Bmatrix}$$

The Spring-Mass model for a 2 DOF system (Continued)

Orthogonality of the Eigen-modes (Normal modes)

$$\{\phi_1\}^T[K]\{\phi_2\} = \{\phi_2\}^T[K]\{\phi_1\} = 0 \qquad \{\phi_1\}^T[K]\{\phi_1\} = k_{11} \qquad \{\phi_2\}^T[K]\{\phi_2\} = k_{22}$$

$$\{\phi_1\}^T[M]\{\phi_2\} = \{\phi_2\}^T[M]\{\phi_1\} = 0 \qquad \{\phi_1\}^T[M]\{\phi_1\} = m_{11} \qquad \{\phi_2\}^T[M]\{\phi_2\} = m_{22}$$

Natural Frequencies (rad/sec)

$$\omega_1 = \sqrt{\frac{k_{11}}{m_{11}}} \qquad \omega_2 = \sqrt{\frac{k_{22}}{m_{22}}}$$

k_{ii} : generalized modal stiffness for mode i
m_{ii}: generalized modal mass for mode i

Estimation of dynamic response for

$$\boxed{[M]\begin{Bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{Bmatrix} + [K]\begin{Bmatrix}x_1\\ x_2\end{Bmatrix} = \begin{Bmatrix}F_1(t)\\ F_2(t)\end{Bmatrix}}$$

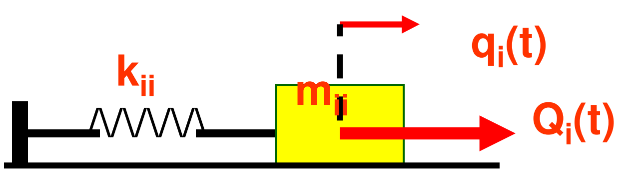
Modal superposition

$$\begin{Bmatrix}x_1\\ x_2\end{Bmatrix} = q_1(t)\{\phi_1\} + q_2(t)\{\phi_2\} = [\{\phi_1\},\{\phi_2\}]\begin{Bmatrix}q_1\\ q_2\end{Bmatrix} = [\phi]\{q\}$$

Pre-multiply equation of motion by modal matrix $[\phi]$

$$[\phi]^T[M]\begin{Bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{Bmatrix} + [\phi]^T[K]\begin{Bmatrix}x_1\\ x_2\end{Bmatrix} = [\phi]^T\begin{Bmatrix}F_1(t)\\ F_2(t)\end{Bmatrix} \Rightarrow [\phi]^T[M][\phi]\begin{Bmatrix}\ddot{q}_1\\ \ddot{q}_2\end{Bmatrix} + [\phi]^T[K][\phi]\begin{Bmatrix}q_1\\ q_2\end{Bmatrix} = \begin{Bmatrix}Q_1(t)\\ Q_2(t)\end{Bmatrix}$$

$$\begin{bmatrix}m_{11} & 0\\ 0 & m_{22}\end{bmatrix}\begin{Bmatrix}\ddot{q}_1\\ \ddot{q}_2\end{Bmatrix} + \begin{bmatrix}k_{11} & 0\\ 0 & k_{22}\end{bmatrix}\begin{Bmatrix}q_1\\ q_2\end{Bmatrix} = \begin{Bmatrix}Q_1(t)\\ Q_2(t)\end{Bmatrix} \rightarrow \boxed{\begin{aligned} & m_{ii}\ddot{q}_i + k_{ii}q_i = Q_i(t) \\ & Q_i(t) = \{\phi_i\}^T\begin{Bmatrix}F_1(t)\\ F_2(t)\end{Bmatrix}\end{aligned}}$$



Modal Decoupling:

The Spring-Mass model: equal stiffnesses and masses

Equation of Motion (free vibration):

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Solution:

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}.\sin(\omega_n t + \varepsilon)$$

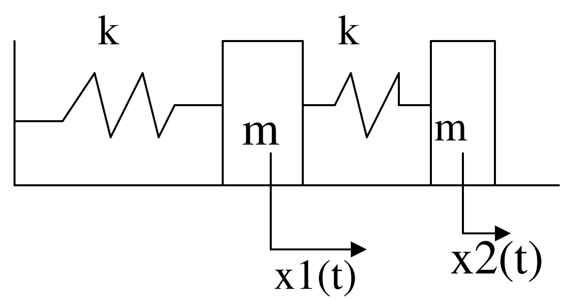
Characteristic Equation:

$$\det\left|\begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} - \omega_n{}^2 \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\right| = 0$$

Natural frequencies (rad/sec) and Eigen-modes

$$\omega_1 = (0.618)\sqrt{\frac{k}{m}} : \qquad \{\phi_1\} = \begin{Bmatrix} 1 \\ 1.618 \end{Bmatrix}$$

$$\omega_2 = (1.618)\sqrt{\frac{k}{m}} : \qquad \{\phi_2\} = \begin{Bmatrix} 1 \\ -0.618 \end{Bmatrix}$$



Observation: The natural frequencies and mode shapes for this particular system are related to the Golden Numbers :

(1+√5)/2=1.618 and (1-√5)/2=-0.618 (!!!)

Stationary Property of the Rayleigh Quotient at the Normal Modes of Vibration

The Spring-Mass model: equal stiffnesses and masses

Solution: $\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} .\sin(\omega t)$

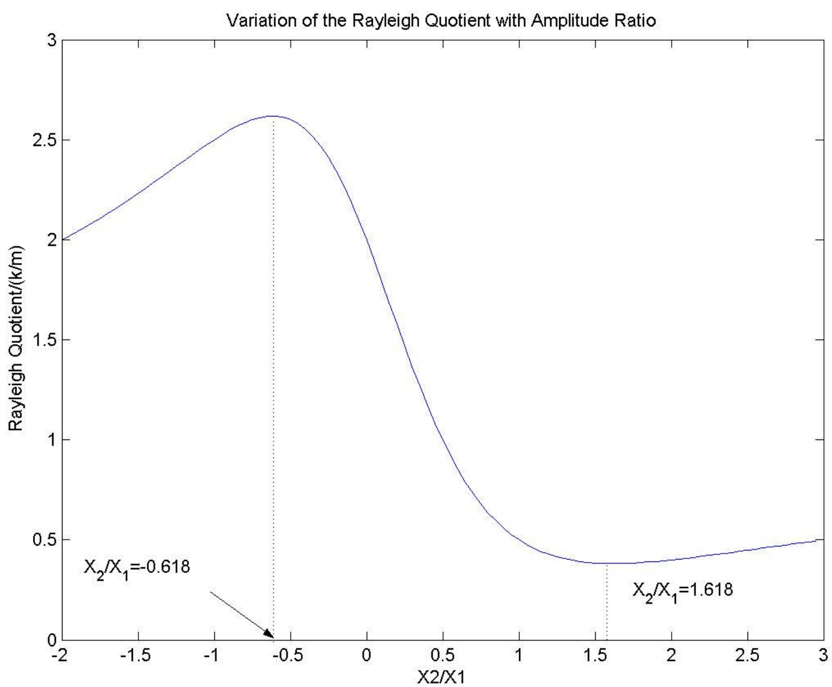
Velocity Vector : $\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \omega \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} .\cos(\omega t)$

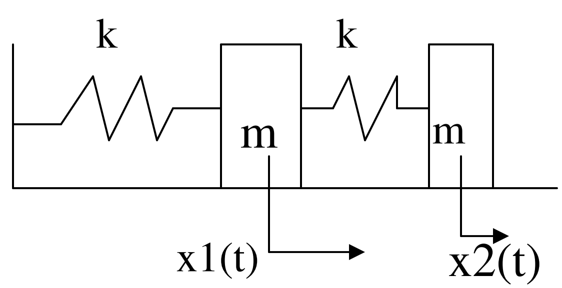
Conservation of Energy: $KE_{\max} = PE_{\max}$

$$\frac{1}{2} m\omega^2 X_1{}^2 + \frac{1}{2} m\omega^2 X_2{}^2 = \frac{1}{2} kX_1{}^2 + \frac{1}{2} k(X_2 - X_1)^2$$

Rayleigh Quotient

$$\omega^2 = \frac{k}{m} \left\{ \frac{\left(\dfrac{X_2}{X_1} - 1 \right)^2 + 1}{\left(\dfrac{X_2}{X_1} \right)^2 + 1} \right\}$$

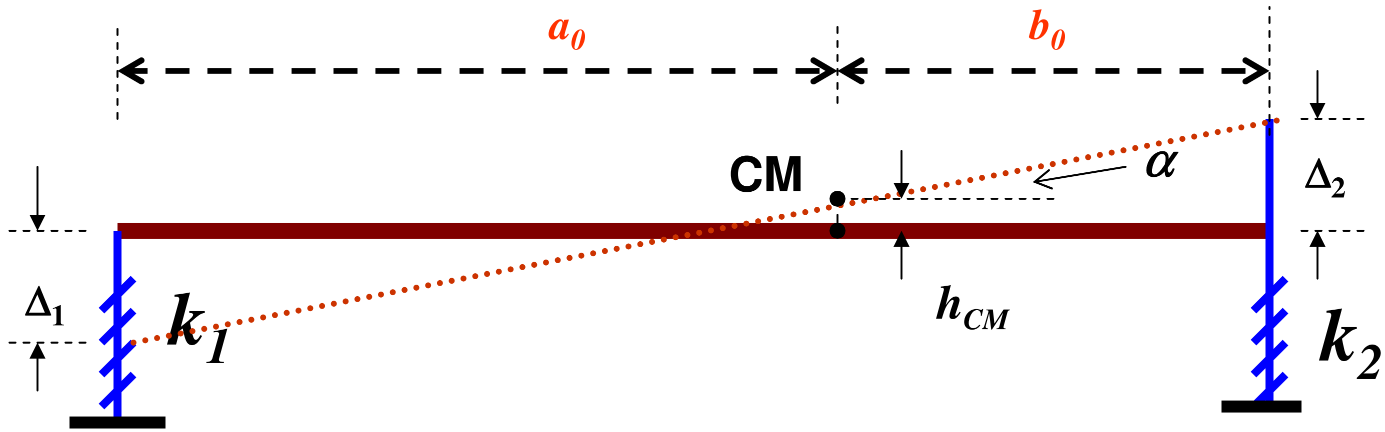




$$\frac{d\omega^2}{d\left(\dfrac{X_2}{X_1}\right)} = 0 \quad : \quad \left(\frac{X_2}{X_1}\right)^2 - \left(\frac{X_2}{X_1}\right) - 1 = 0 \qquad \left(\frac{X_2}{X_1}\right) = 1.618, \quad -0.618$$

Observation: The Rayleigh Quotient is extremum (stationary) at the normal modes.

Example 3 A rigid car chassis as a 2 DOF system



Case 1: Point of reference for displacements : Center of mass, CM

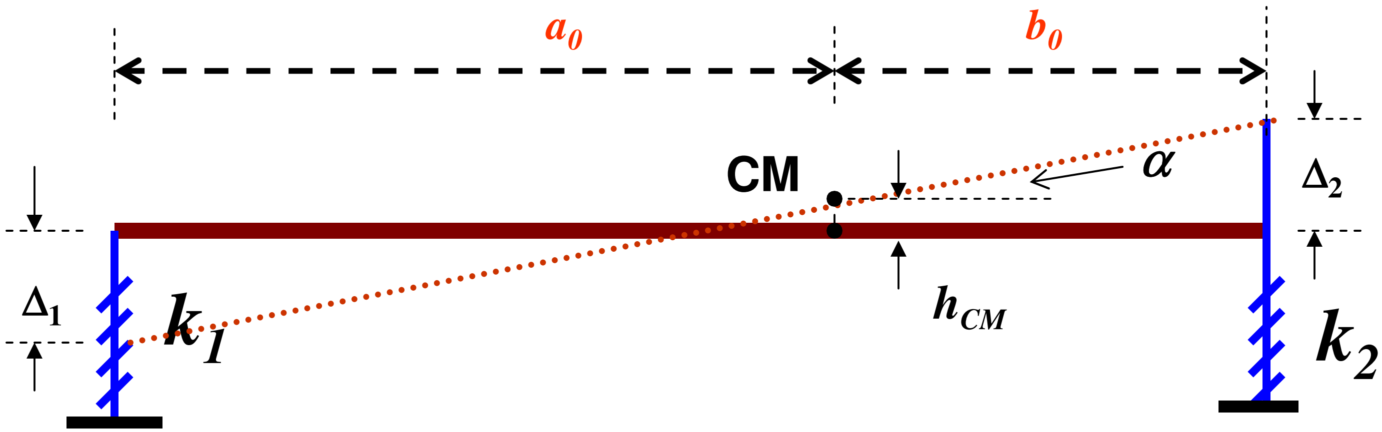
Expressions for kinetic energy T, potential energy V and the Lagrangian L are

$$T = \frac{1}{2} m\dot{h}_{CM}{}^2 + \frac{1}{2} I_{CM}\dot{\alpha}^2$$

$$V = \frac{1}{2} k_1 \Delta_1{}^2 + \frac{1}{2} k_2 \Delta_2{}^2 = \frac{1}{2} k_1 \{a_0\alpha - h_{CM}\}^2 + \frac{1}{2} k_2 \{b_0\alpha + h_{CM}\}^2$$

$$L = T - V = \frac{1}{2} m\dot{h}_{CM}{}^2 + \frac{1}{2} I_{CM}\dot{\alpha}^2 - \frac{1}{2} k_1 \{a_0\alpha - h_{CM}\}^2 - \frac{1}{2} k_2 \{b_0\alpha + h_{CM}\}^2$$

Example 3 A rigid car chassis as a 2 DOF system (Continued)



Case 1: Point of reference for displacements : Center of mass, CM

Lagrange's Equations $\quad \frac{d}{dt}\frac{\partial L}{\partial \dot{h}} - \frac{\partial L}{\partial h} = 0 \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha}} - \frac{\partial L}{\partial \alpha} = 0$

Equations of free vibration in matrix form

$$\boxed{\begin{bmatrix} m & 0 \\ 0 & I_{CM} \end{bmatrix} \begin{Bmatrix} \ddot{h}_{CM} \\ \ddot{\alpha} \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -(k_1 a_0 - k_2 b_0) \\ -(k_1 a_0 - k_2 b_0) & k_1 a_0{}^2 + k_2 b_0{}^2 \end{bmatrix} \begin{Bmatrix} h_{CM} \\ \alpha \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}}$$

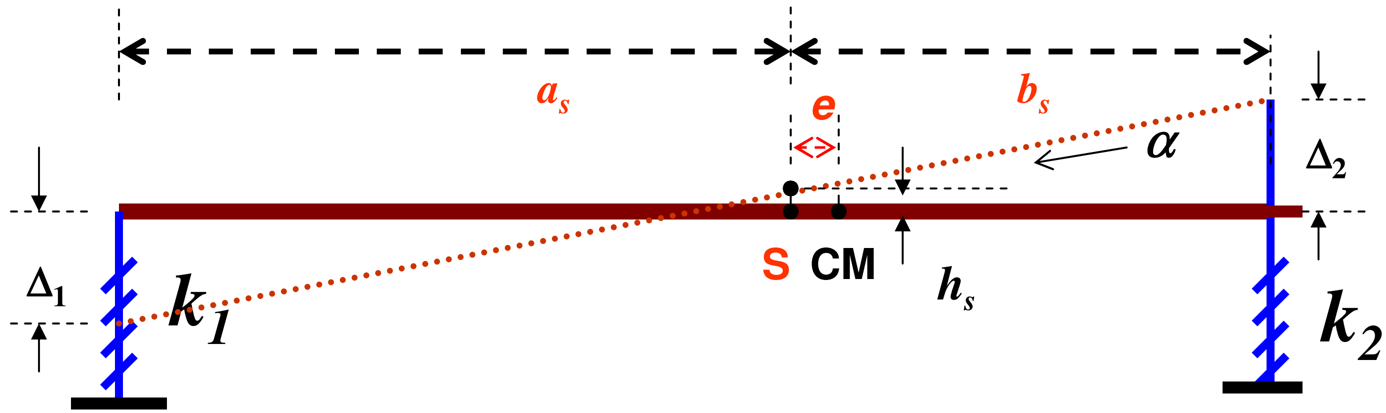
Note:

For displacements at the center of mass, the system equations are inertially decoupled. But the system equations are in general stiffness coupled.

If however, the spring stiffness values be in the ratio then the system is also stiffness decoupled.

$$\boxed{\frac{k_1}{k_2} = \frac{b_0}{a_0} \quad \Rightarrow \quad k_1 a_0 - k_2 b_0 = 0}$$

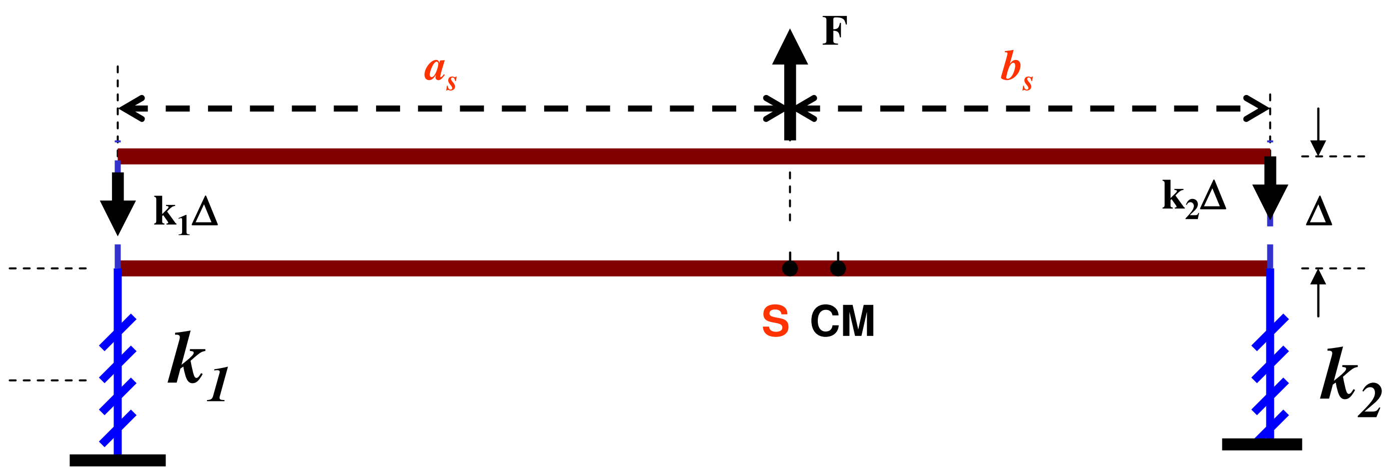
Example 3 A rigid car chassis as a 2 DOF system (Continued)



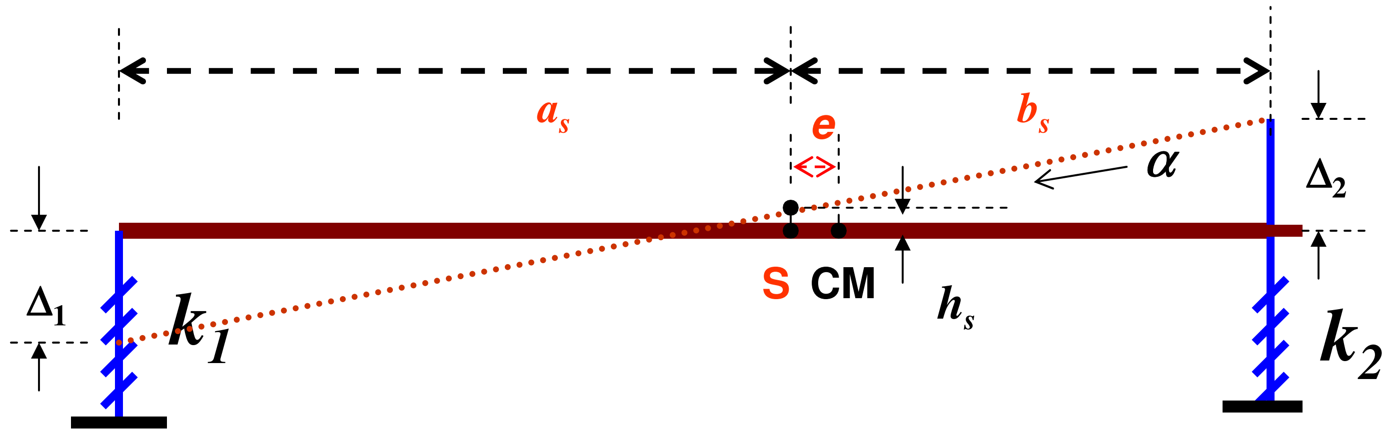
Case 2: A special point of reference S for displacements so that

$$\frac{b_s}{a_s} = \frac{k_1}{k_2} \quad \Rightarrow \quad k_1 a_s - k_2 b_s = 0 \longrightarrow \boxed{(k_1\Delta)a_s - (k_2\Delta)b_s = 0}$$

This means that under an external force F acting the point S, one can have pure translation (no rotation) of the rigid chassis without any rotation (equal and opposite moments are induced by the springs about point S).



Example 3 A rigid car chassis as a 2 DOF system (Continued)



Case 2: A special point of moment balance S for displacements

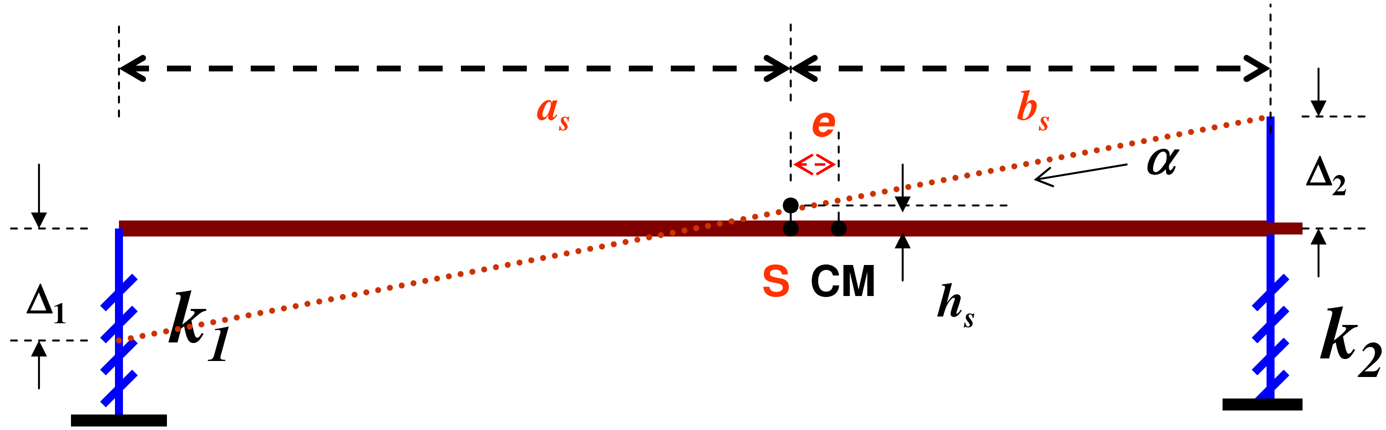
$$T = \frac{1}{2} m \dot{h}_{CM}^{\ 2} + \frac{1}{2} I_{CM} \dot{\alpha}^2 = \frac{1}{2} m\left(\dot{h}_S + e\dot{\alpha}\right)^2 + \frac{1}{2} I_{CM} \dot{\alpha}^2 = \frac{1}{2} m \dot{h}_s^{\ 2} + me\dot{h}_s\dot{\alpha} + \frac{1}{2} I_S \dot{\alpha}^2$$

$$I_s = I_{CM} + me^2$$

$$V = \frac{1}{2} k_1 \Delta_1^{\ 2} + \frac{1}{2} k_2 \Delta_2^{\ 2} = \frac{1}{2} k_1 \{a_s\alpha - h_s\}^2 + \frac{1}{2} k_2 \{b_s\alpha + h_s\}^2$$

$$L = T - V = \frac{1}{2} m \dot{h}_s^{\ 2} + me\dot{h}_s\dot{\alpha} + \frac{1}{2} I_s \dot{\alpha}^2 - \frac{1}{2} k_1 \{a_s\alpha - h_s\}^2 - \frac{1}{2} k_2 \{b_s\alpha + h_s\}^2$$

Example 3 A rigid car chassis as a 2 DOF system (Continued)



Case 2: A special point of moment balance S for displacements

Lagrange's Equations $\quad \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{h}} - \dfrac{\partial L}{\partial h} = 0 \quad$ **&** $\quad \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\alpha}} - \dfrac{\partial L}{\partial \alpha} = 0$

Apply the condition $\quad \boxed{\dfrac{b_s}{a_s} = \dfrac{k_1}{k_2} \quad \Rightarrow \quad k_1 a_s - k_2 b_s = 0}$

Equations of free vibration in matrix form

$$\boxed{\begin{bmatrix} m & me \\ me & I_{CM} \end{bmatrix} \begin{Bmatrix} \ddot{h}_s \\ \ddot{\alpha} \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & 0 \\ 0 & k_1 a_s^{\,2} + k_2 b_s^{\,2} \end{bmatrix} \begin{Bmatrix} h_s \\ \alpha \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}}$$

Note:
For displacements at the special point S for balance of moments from spring forces <u>the system equations</u> are <u>stiffness decoupled</u>.
But the system equations are in general <u>inertially coupled</u>.
If CM and S coincide then system is both stiffness wise and inertially decoupled.

2.4 Constrained Systems; Lagrange's Multipliers

A system of '*n*' degrees of freedom (DOFs) is *holonomic* if all its DOFs are mutually independent.

A system is *non-holonomic* when some of its '*n*' degrees of freedom (DOFs) are dependent on the others. This means that it has some '*r*' constraints so that the actual number of its independent degrees of freedom is (*n-r*).

We need to find the stationary points of the functional

$$I = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)dt \qquad i = 1,2,\ldots n$$

Subject to *r* constraints

$$f_j(q_i) = constant \qquad \textbf{or} \qquad \int_{t_1}^{t_2} f_j(q_i, \dot{q}_i, t)dt = constant \qquad j = 1,2\ldots r$$

Procedure:

Modify the Lagrangian

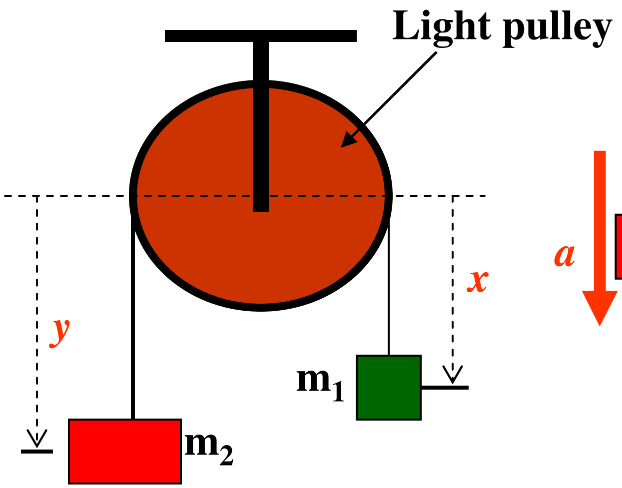
$$L^* = L + \sum_{j=1}^{r} \lambda_j f_j$$

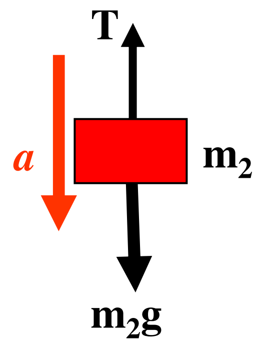
Equation of motion can be had from $\delta I = 0 \quad where \quad I = \int_{t_1}^{t_2} L^*(q_i, \dot{q}_i, t)dt$

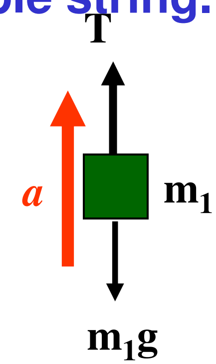
Modified Lagrange's Equations: $\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \sum_{j=1}^{r} \lambda_j \frac{\partial f_j}{\partial q_i} = 0 \quad or \quad = Q_i : Non-conservative \quad force}$

Example 4 The Atwood Machine

Find the expressions for accelerations of the two unequal masses suspended from the machine by an in-extensible string.







Method 1: Using Newton's Law

$$m_1 a = T - m_1 g \text{.............}(i)$$

$$m_2 a = m_2 g - T \text{...........}(ii)$$

$$(i)+(ii)$$

$$(m_1 + m_2)a = (m_2 - m_1)g$$

$$a = \frac{(m_2 - m_1)}{(m_2 + m_1)} g \qquad m_2 \geq m_1$$

$$\ddot{x} = -a \uparrow \qquad \ddot{y} = a \downarrow$$

Method 2: Using Lagrange's Equations

$$x + y + \pi R = \ell = fixed \quad length \quad of \quad string$$

$$\therefore\ y = -x - (\pi R + \ell) = -x - constant$$

$$and \quad \dot{y} = -\dot{x} \quad and \quad \ddot{y} = -\ddot{x}$$

$$KE = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{y}^2 = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 (-\dot{x}^2) = \frac{1}{2}(m_1 + m_2)\dot{x}^2$$

$$V = -m_1 g x - m_2 g y = -m_1 g x - m_2 g \{-x - (\pi R + \ell)\} = -(m_1 - m_2).gx + (constant)_1$$

$$L = KE - V$$

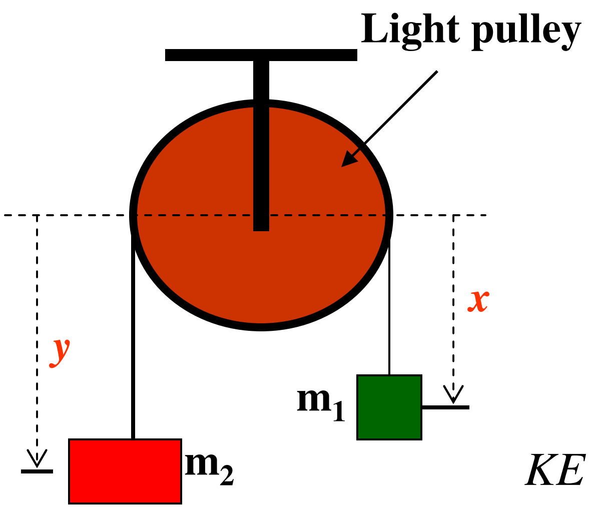
$$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 - [-(m_1 - m_2).gx + (constant)_1]$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \quad \rightarrow \quad (m_1 + m_2)\ddot{x} - (m_1 - m_2)g = 0$$

$$\ddot{x} = \frac{(m_1 - m_2)}{(m_1 + m_2)} g \qquad \ddot{y} = -\ddot{x} = \frac{(m_2 - m_1)}{(m_1 + m_2)} g$$

Example 4 The Atwood Machine (continued)

Method 3: Using Lagrange's Equations with Lagrange's Multipliers



Constraint Equation

$$x + y + \pi R = \ell = fixed \quad length \quad of \quad string$$

$$\boxed{f(x, y) = x + y = constant} \rightarrow \boxed{\ddot{y} = -\ddot{x}}$$

One equation from constraint

Lagrangian L

$$KE = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{y}^2 \qquad V = -m_1 g x - m_2 g y$$

$$L = KE - V = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{y}^2 + (m_1 g x + m_2 g y)$$

Lagrange's Equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} + \lambda\left(\frac{\partial f}{dx}\right) = 0 \quad \rightarrow \quad m_1 \ddot{x} - m_1 g + \lambda .1 = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} + \lambda\left(\frac{\partial f}{\partial y}\right) = 0 \quad \rightarrow \quad m_2 \ddot{y} - m_2 g + \lambda .1 = 0$$

Two more equations from Lagrange Equations

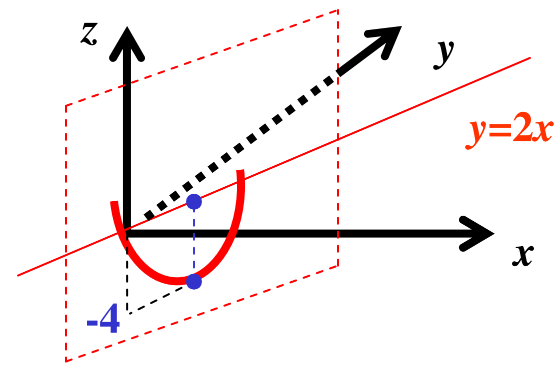
Solving all three equations:

$$\ddot{x} = \frac{(m_1 - m_2)}{(m_1 + m_2)} g \qquad \ddot{y} = -\ddot{x} = \frac{(m_2 - m_1)}{(m_1 + m_2)} g$$

$$\lambda = m_1 (g - \ddot{x}) = T$$

The Lagrange Multiplier is the constraint force, *viz.* tension T is string

Example 5 A simple constrained minimal problem



Minimize the function

$$z(x, y) = x^2 + y^2 - 2xy - 4x$$

Subject to the constraint

$$y - 2x = 0$$

Using Direct Substitution	Using Lagrange's Multiplier	Using Penalty Method (This type is used to describe many FE formulations)
$y = 2x$ $\therefore z = x^2 + y^2 - 2xy - 4x = x^2 - 4x$ $\frac{\partial z}{\partial x} = 2x - 4$ *At stationary point* $\frac{\partial z}{\partial x} = 0 \quad \Rightarrow \quad (x_m, y_m) = (2,4)$ $z_{\min} = x_m{}^2 - 4x_m = -4$	$z_1 = x^2 + y^2 - 2xy - 4x + \lambda(y - 2x)$ *At stationary point (minima)* $\frac{\partial z_1}{\partial x} = 2x - 2y - 4 - 2\lambda = 0$ $\frac{\partial z_1}{\partial y} = -2x + 2y + \lambda = 0$ $\frac{\partial z_1}{\partial \lambda} = -2x + y = 0$ *Solving;* $x_m = 2, \quad y_m = 4 \quad (z_1)_m = -4$	$z_2 = x^2 + y^2 - 2xy - 4x + \kappa(y - 2x)^2$ *At stationary point (minima)* $\frac{\partial z_2}{\partial x} = 2x - 2y - 4 - 4\kappa(y - 2x) = 0$ $\frac{\partial z_2}{\partial y} = -2x + 2y + 2\kappa(y - 2x) = 0$ *Solving;* $x_m = 2\left(1 + \frac{1}{\kappa}\right), \quad y_m = 2\left(2 + \frac{1}{\kappa}\right)$ *As* $\kappa \to \infty$, $x_m \to 2, \quad y_m \to 4 \quad (z_2)_m \to -4$

Lecture 2

Variational Principles in Computational Solid Mechanics

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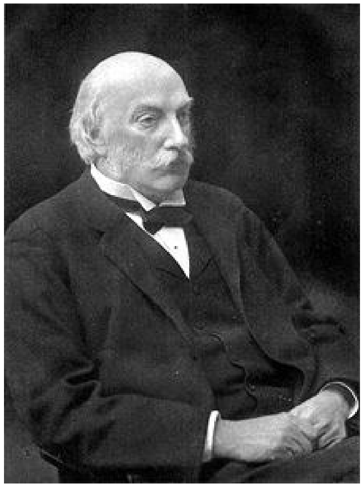
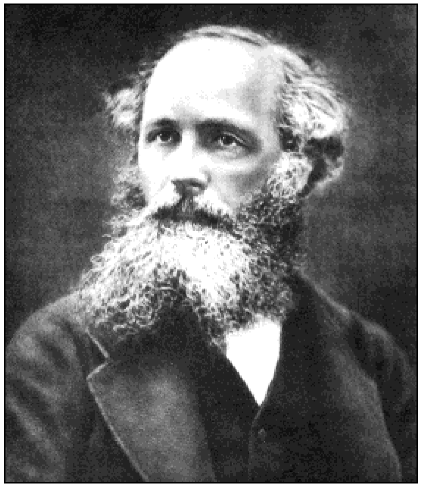
Lecture 2

Chapter 3

Weighted Residual (Galerkin's) method and the principle of minimum potential energy.



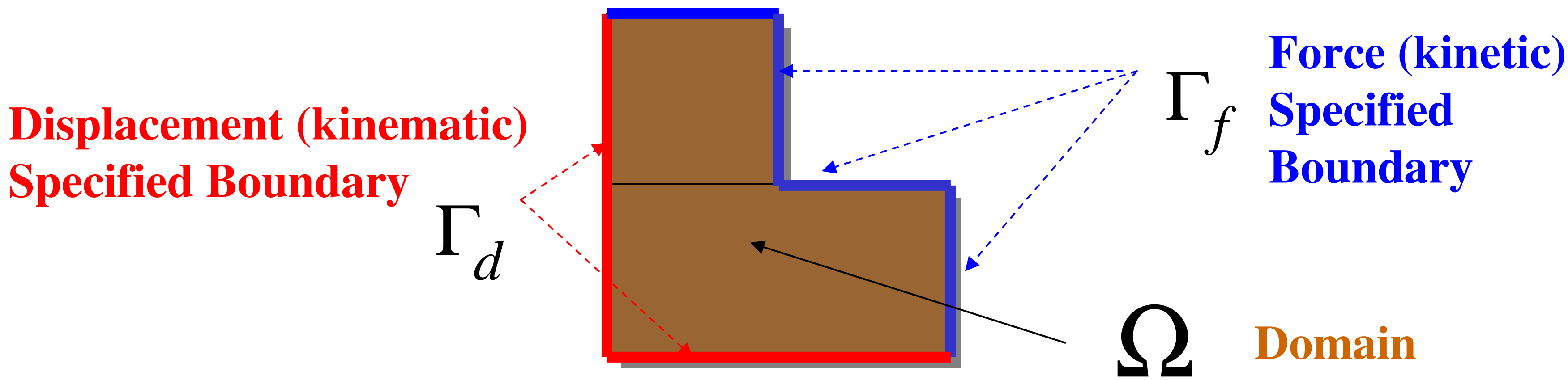
Lord Rayleigh



James Clerk Maxwell

3.1 The self adjoint operator

Consider a differential equation: $\mathscr{L}(u) = f$



Use a trial function v as virtual displacement

$$\int_{\Omega} v.\mathscr{L}(u).\,d\Omega = \int_{\Omega} u.\mathscr{L}^{*}(v)d\Omega + \int_{\Gamma=\Gamma_d+\Gamma_f} \{d(v)f(u) - d(u)f^{*}(v)\}d\Gamma$$

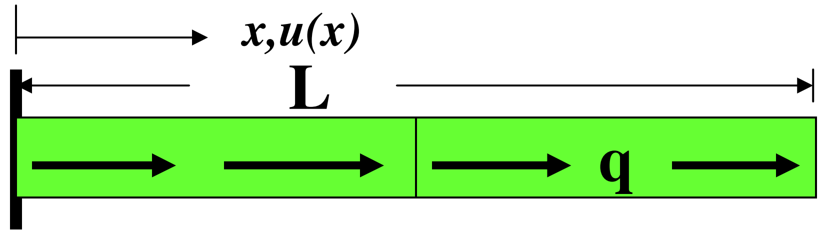
$$\boxed{If \quad \int_{\Omega} v.\mathscr{L}(u).\,d\Omega = \int_{\Omega} u.\mathscr{L}^{*}(v)d\Omega \quad \Rightarrow \mathscr{L} \quad is \quad self-adjoint \quad operator} \quad \textbf{(3.1)}$$

This is the consequence of the Reciprocal Theorem (valid for conservative systems only).

A conservative system is characterized by a differential equation of self- adjoint operator. A conservative system has a symmetric stiffness matrix (or compliance matrix). The energy of a conservative system is independent of changing order of loads.

Example 1. **Show that the linear differential operator for the axially loaded bar is self-adjoint.**

$$\boxed{-\frac{d}{dx}\left(EA\frac{du}{dx}\right)=q} \qquad \mathscr{L}=-\frac{d}{dx}\left(EA\frac{d}{dx}\right)$$



$$\int_\Omega v.\mathscr{L}(u).d\Omega=\int_0^L v\left[-\frac{d}{dx}\left(EA\frac{du}{dx}\right)\right]dx=\left[v\left(-EA\frac{du}{dx}\right)\right]_{x=0}^{x=L}+\int_0^L\frac{dv}{dx}\left(EA\frac{du}{dx}\right)dx$$

$$=\left[v\left(-EA\frac{du}{dx}\right)\right]_{x=0}^{x=L}+\int_0^L\frac{du}{dx}\left(EA\frac{dv}{dx}\right)dx=\left[v\left(-EA\frac{du}{dx}\right)\right]_{x=0}^{x=L}+\left[\left(EA\frac{dv}{dx}\right).u\right]_{x=0}^{x=L}+\int_0^L u\left[-\frac{d}{dx}\left(EA\frac{dv}{dx}\right)\right]dx$$

$$=\int_0^L u\left[-\frac{d}{dx}\left(EA\frac{dv}{dx}\right)\right]dx+\left[v\left(-EA\frac{du}{dx}\right)\right]_{x=0}^{x=L}-\left[u\left(-EA\frac{dv}{dx}\right)\right]_{x=0}^{x=L}$$

$$\boxed{0}$$

$$\boxed{\int_\Omega v.\mathscr{L}(u).d\Omega=\int_\Omega u.\mathscr{L}^*(v)d\Omega+\{vf(u)-uf^*(v)\}_{\Gamma=\Gamma_d+\Gamma_f}}$$

$$\boxed{\int_0^L v\left[-\frac{d}{dx}\left(EA\frac{du}{dx}\right)\right]dx=\int_0^L u\left[-\frac{d}{dx}\left(EA\frac{dv}{dx}\right)\right]dx}$$

Reciprocal Theorem

3.2 Weighted Residual Method

for a system of differential equation of equilibrium with self-adjoint operator

$$\int_\Omega \delta u^h . f \, d\Omega$$

Consider a differential equation: $\mathcal{L}(u) = f$

Use an admissible trial function δu^h as virtual displacement, and *u* is the analytical solution

0

$$\int_\Omega \delta u^h . \{\mathcal{L}(u) - f\} d\Omega = a(\delta u^h, u) - (\delta u^h, f) - \left[\delta u^h . R\right]_B$$

B is the domain boundary. Boundary terms vanish when either $\delta u^h = 0$ (specified kinematic boundary condition) or reaction R=0 at boundary B.

$$\boxed{\begin{aligned} 0 &= a(\delta u^h, u) - (\delta u^h, f) \\ \Rightarrow a(\delta u^h, u) &= (\delta u^h, f) \end{aligned}} \qquad \textbf{(3.2)}$$

For a self-adjoint operator, *a*(u,v) is symmetric bilinear functional

$$\boxed{\int_\Omega \delta u^h . \mathcal{L}(u) . d\Omega = \int_\Omega u . \mathcal{L}^*(\delta u^h) d\Omega \quad \Rightarrow a(\delta u^h, u) = a(u, \delta u^h)} \qquad \textbf{(3.3)}$$

Case 1

If the trial function is a variation of the exact displacement

$$\boxed{\delta u^h = \delta u}$$

$$a(\delta u, u) = (\delta u, f)$$

$$\delta \frac{1}{2} a(u,u) = \delta W$$

Here $\boxed{\frac{1}{2} a(u,u)}$ **is the analytical strain energy *U* of the system**

and δW is the virtual work done by the external forces.

If we define a system potential energy (for a conservative system)

$$\boxed{\Pi = \frac{1}{2} a(u,u) - (u, f) = U - W} \tag{3.4}$$

Then equilibrium condition : $\delta \Pi = 0$

Principle of stationary potential energy

Case 2

If the exact function (of differential equation) is replaced by the trial function $u^h(x) = \sum_{i=1}^{N} a_i \phi_i(x)$ $\quad \delta u^h(x) = \sum_{i=1}^{N} (\delta a_i).\phi_i(x)$

Thus $\boxed{\mathcal{L}(u^h) - f = e}$ ***e*=residual**

***Weighted residual statement* is**

$$\boxed{\int_\Omega \delta u^h.\{\mathcal{L}(u^h) - f\} d\Omega = 0}$$

$$\boxed{\delta a_i \int_\Omega \phi_j.\left\{\mathcal{L}\left(\sum_{i=1}^{N} a_i \phi_i(x)\right) - f\right\} d\Omega = 0 \qquad j = 1,2 \ldots N} \quad \textbf{(3.5)}$$

$$a(\delta u^h, u^h) = (\delta u^h, f)$$

$$\delta \frac{1}{2} a(u^h, u^h) = \delta W$$

Galerkin's Method

Rayleigh-Ritz Method

If we define a system potential energy (for a conservative system)

$$\boxed{\Pi = \frac{1}{2} a(u^h, u^h) - (u^h, f) = U - W} \qquad \textbf{(3.6)}$$

Then equilibrium condition : *δΠ*=0

Rayleigh Ritz Method (RRM) or Galerkin's Method (GM); Which should we use?

For conservative systems, the equations of equilibrium (with any given approximate function for displacement) determined by both Rayleigh- Ritz and Galerkin's Methods are identical.

For non-conservative systems, we cannot adopt the Rayleigh-Ritz Method (RRM), since the reciprocal theorem (symmetry; equation (3.1)) does not hold, and consequently we cannot define the potential energy functional π for the system.

Such non-conservative systems are described by differential equations with Non-self-adjoint operators.

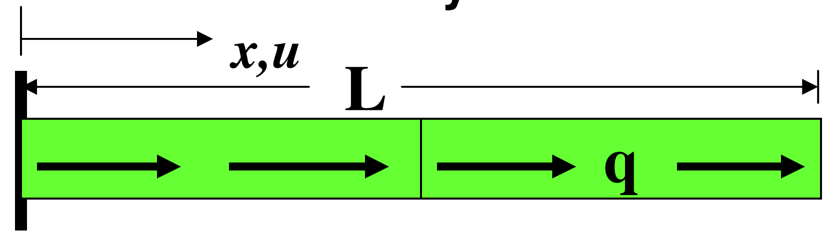
However, the virtual work principle remains valid even for non-conservative systems. Thus Galerkin's method (GM) of weighted residuals can be applied even to non-conservative systems to establish the finite element equations (force-displacement relation) of equilibrium from the corresponding differential equations of non-self-adjoint differential operators .

Message: Use RRM or GM for conservative systems

Use only GM for non-conservative systems

Example 2. Weighted residual statements for the axially loaded bar.

$$\boxed{-\frac{d}{dx}\left(EA\frac{du}{dx}\right)=q} \qquad \mathcal{L}=-\frac{d}{dx}\left(EA\frac{d}{dx}\right)$$



$$\int_\Omega \delta u^h.\{\mathcal{L}(u)\text{ - }q\}d\Omega=\int_0^L\delta u^h\left[-\frac{d}{dx}\left(EA\frac{du}{dx}\right)-q\right]dx$$

$$0=\cancelto{0}{\left[\delta u^h\left(-EA\frac{du}{dx}\right)\right]_{x=0}^{x=L}}+\int_0^L\frac{d\delta u^h}{dx}\left(EA\frac{du}{dx}\right)dx-\int_0^L\delta u^h q.dx$$

$$0=a(\delta u^h,u)-(\delta u^h,q)$$

$$a(\delta u^h,u)=\int_0^L\frac{d\delta u^h}{dx}\left(EA\frac{du}{dx}\right)dx$$

$$\frac{1}{2}a(u,u)=\frac{1}{2}\int_0^L\left(\frac{du}{dx}\right)EA\left(\frac{du}{dx}\right)dx$$

$$\frac{1}{2}a(u^h,u^h)=\frac{1}{2}\int_0^L\left(\frac{du^h}{dx}\right)EA\left(\frac{du^h}{dx}\right)dx$$

Weighted residual statement **is**

$$\boxed{\delta a_i\int_\Omega\phi_j.\left\{\mathcal{L}\left(\sum_{i=1}^{N}a_i\phi_i(x)\right)\text{ - }q\right\}d\Omega=0 \qquad j=1,2\ldots N}$$

→ ***Galerkin's Method***

$$\boxed{\delta\Pi=0}$$

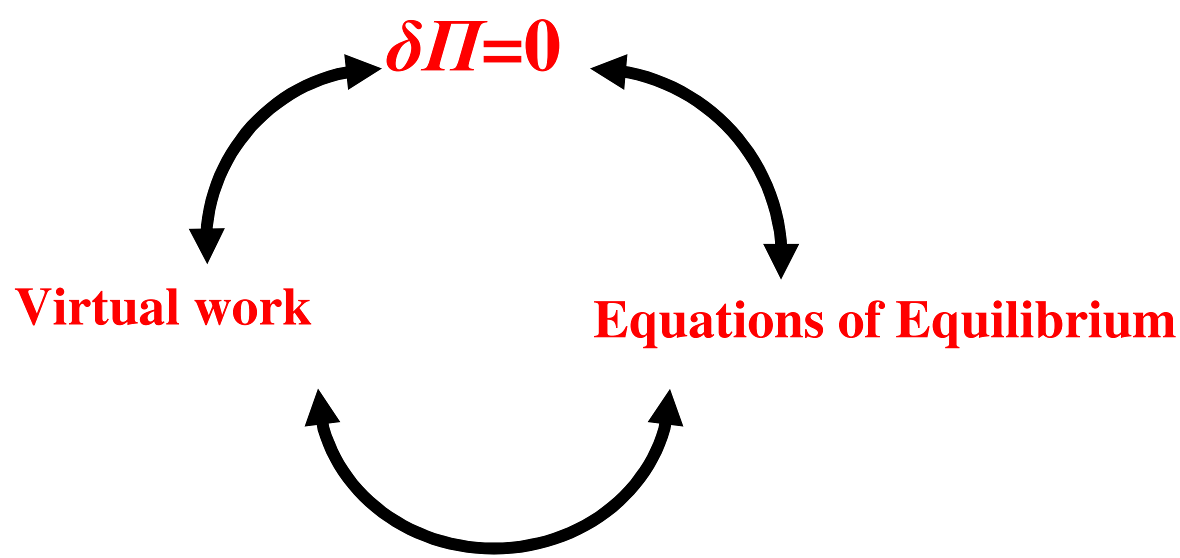
→ ***Rayleigh-Ritz Method***

$$\Pi=\frac{1}{2}a(u^h,u^h)-(u^h,q)=\frac{1}{2}\int_0^L\left(\frac{du^h}{dx}\right)EA\left(\frac{du^h}{dx}\right)dx-\int_0^L u^h q.dx \qquad \Pi=U-W$$

3.3 The principle of minimum potential energy and equations of equilibrium

For a conservative (self-adjoint) system, we can identify a potential energy functional *Π*, so that at equilibrium, any admissible virtual displacement leads to

$$\delta \Pi = 0$$




THE TIMES OF INDIA, BANGALORE
TUESDAY, JULY 28, 2009

SHORT CUTS

AP

LOVE ACROSS SPECIES: Three lion cubs and a tiger cub (top) play with their surrogate mother, a dog, in Hefei, China. The dog has been feeding the cubs since June after their biological mothers abandoned them

Incompatible elements in harmony (equilibrium)

Compatible elements not in harmony

The principle of minimum potential energy
(for searching equilibrium)

"Of all the possible admissible displacements (satisfying the kinematic boundary conditions) the one that corresponds to equilibrium (of forces) is the one that makes the potential energy stationary."

(*i.e.* any virtual displacement about equilibrium (of forces) brings about a vanishing first variation of potential Energy)

$$\delta \Pi = 0 \text{ for } \delta u \tag{3.7}$$

A stable equilibrium makes the potential energy a minimum

The principle of stationary complementary energy (for searching compatibility)

"Of all the possible admissible forces (in equilibrium) the one that corresponds to displacement compatibility is the one that makes the complementary energy stationary."

(*i.e.* any virtual change of forces in equilibrium brings about a vanishing first variation of complementary energy).

$$\delta\Pi^{*}=0 \text{ for } \delta F \qquad (3.8)$$

Complementary energy and its variation is

$$\Pi^{*}=U^{*}-W \qquad \delta\Pi^{*}=\delta U^{*}-\delta W(\text{for } \delta F)$$

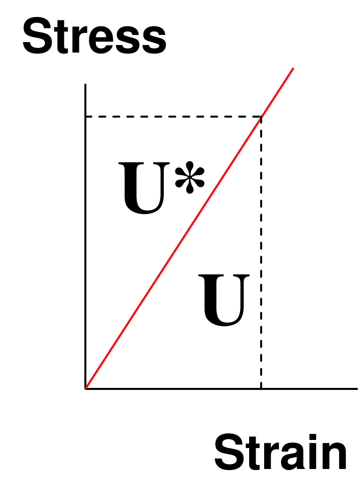
3.4 Castigliano's Theorems

The principle of minimum potential energy (for searching equilibrium)

$$\delta\Pi = \delta U - \sum_{i=}^{n} P_i \delta u_i = 0$$

$$P_i = \frac{\partial U}{\partial u_i} \qquad \textbf{(3.9)}$$

The principle of stationary complementary energy (for searching compatibility)



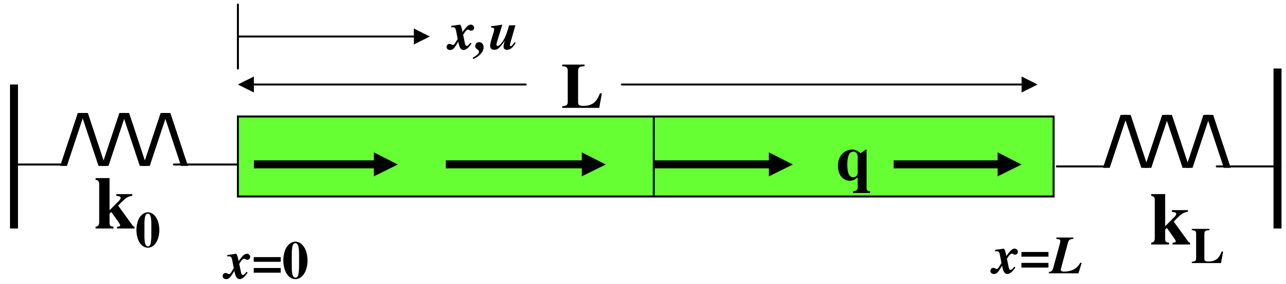
$$\delta\Pi^* = \delta U^* - \sum u_i \delta P_i = 0$$

$$u_i = \frac{\partial U^*}{\partial P_i} \qquad \textbf{(3.10)}$$

For a linear structure, $U^* = U$

Complementary energy/volume equals strain energy/volume

Example 3. **Derive the equations of equilibrium and boundary conditions for the axially loaded bar with spring supported ends.**



Potential energy functional $\Pi = U - W$

$$\Pi = \left[\frac{1}{2}\int_0^L \left(\frac{du}{dx}\right) EA \left(\frac{du}{dx}\right) dx + \frac{1}{2} k_0 {u_0}^2 + \frac{1}{2} k_L {u_L}^2 \right] - \int_0^L uq.dx$$

$$\delta\Pi = \int_0^L \delta\left(\frac{du}{dx}\right).EA\left(\frac{du}{dx}\right)dx + + k_0 u_0.\delta u_0 + k_L u_L.\delta u_L - \int_0^L \delta u.q.dx$$

$$\delta\Pi = \left[EA\left(\frac{du}{dx}\right).\delta u\right]_{x=0}^{x=L} + k_0 u_0.\delta u_0 + k_L u_L.\delta u_L - \int_0^L \delta u\left[\frac{d}{dx}\left(EA\frac{du}{dx}\right)\right] - \int_0^L \delta u.q.dx$$

$$\delta\Pi = \left[\left(-EA\frac{du}{dx} + k_0 u_0\right)_{x=0} \delta u_0 + \left(EA\frac{du}{dx} + k_L u_L\right)_{x=L} \delta u_L\right] + \int_0^L \delta u\left[-\frac{d}{dx}\left(EA\frac{du}{dx}\right) - q\right]dx$$

Equilibrium $\delta\Pi=0$

$$0=\left[\left(-EA\frac{du}{dx}+k_0u_0\right)_{x=0}\delta u_0+\left(EA\frac{du}{dx}+k_Lu_L\right)_{x=L}\delta u_L\right]+\int_0^L\delta u\left[-\frac{d}{dx}\left(EA\frac{du}{dx}\right)-q\right]dx$$

For non-trivial arbitrary, yet admissible virtual displacements

$$\boxed{-\frac{d}{dx}\left(EA\frac{du}{dx}\right)-q=0}$$

Differential equation for axially loaded bar

(3.11)

$$\begin{array}{llll} Either & \delta u_0=0 & or & \left(-EA\dfrac{du}{dx}+k_0u_0\right)_{x=0}=0 \\ Either & \delta u_L=0 & or & \left(EA\dfrac{du}{dx}+k_Lu_L\right)_{x=L}=0 \end{array}$$

Boundary conditions at the ends x=0 & x=L

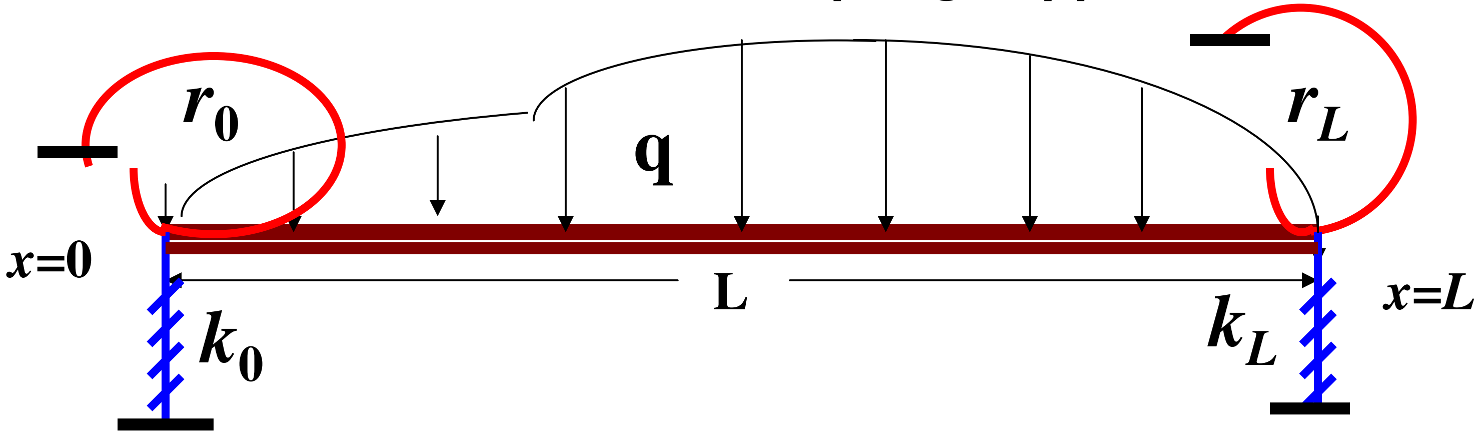
(3.12)

For specified kinematic boundary conditions

Force boundary conditions with spring ends

Force boundary conditions with free ends $\left(-EA\dfrac{du}{dx}\right)_{x=0}=0,\quad \left(EA\dfrac{du}{dx}\right)_{x=L}=0$

Example 4. **Derive the equations of equilibrium and boundary conditions for the Euler beam with spring supported ends.**



Potential energy functional **Π=U-W**

$$\Pi=\left[\frac{1}{2}\int_0^L\left(\frac{d^2w}{dx^2}\right)EI\left(\frac{d^2w}{dx^2}\right)dx+\frac{1}{2}k_0{w_0}^2+\frac{1}{2}k_L{w_L}^2+\frac{1}{2}r_0\left(\frac{dw}{dx}\right)^2_{x=0}+\frac{1}{2}r_L\left(\frac{dw}{dx}\right)^2_{x=L}\right]-\int_0^L wq.dx$$

$$\delta\Pi=\int_0^L\delta\left(\frac{d^2w}{dx^2}\right).EI\left(\frac{d^2w}{dx^2}\right)dx-\int_0^L\delta wq.dx$$

$$+k_0w_0.\delta w_0+k_Lw_L.\delta w_L+r_0\left(\frac{dw}{dx}\right)_{x=0}\delta\left(\frac{dw}{dx}\right)_{x=0}+r_L\left(\frac{dw}{dx}\right)_{x=L}\delta\left(\frac{dw}{dx}\right)_{x=L}$$

$$\delta\Pi = \int_0^L \delta w.\left[\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) - q\right]dx$$

$$+\left[\left(-EI\frac{d^2w}{dx^2} + r_0\frac{dw}{dx}\right)_{x=0}\delta\left(\frac{dw}{dx}\right)_{x=0} + \left(EI\frac{d^2w}{dx^2} + r_L\frac{dw}{dx}\right)_{x=L}\delta\left(\frac{dw}{dx}\right)_{x=0}\right]$$

$$+\left[\left(\frac{d}{dx}EI\frac{d^2w}{dx^2} + k_0 w\right)_{x=0}\delta w_{x=0} + \left(-\frac{d}{dx}EI\frac{d^2w}{dx^2} + k_L w\right)_{x=L}\delta w_{x=L}\right]$$

For non-trivial, arbitrary, yet admissible virtual displacements,

Equilibrium $\delta\Pi = 0$.

$$\boxed{\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) - q = 0}$$

Differential equation for the Euler beam

(3.13)

$$\boxed{\begin{array}{llll}
\text{Either} & \delta\left(\frac{dw}{dx}\right)_{x=0} = 0 & \text{or} & \left(-EI\frac{d^2w}{dx^2} + r_0\frac{dw}{dx}\right)_{x=0} = 0 \\
\text{Either} & \delta\left(\frac{dw}{dx}\right)_{x=L} = 0 & \text{or} & \left(EI\frac{d^2w}{dx^2} + r_L\frac{dw}{dx}\right)_{x=L} = 0 \\
\text{Either} & \delta w_0 = 0 & \text{or} & \left(\frac{d}{dx}EI\frac{d^2w}{dx^2} + k_0 w\right)_{x=0} = 0 \\
\text{Either} & \delta w_L = 0 & \text{or} & \left(-\frac{d}{dx}EI\frac{d^2w}{dx^2} + k_L w\right)_{x=L} = 0
\end{array}}$$

Boundary conditions at the ends x=0 & x=L

(3.14)

3.5 Constrained media problems

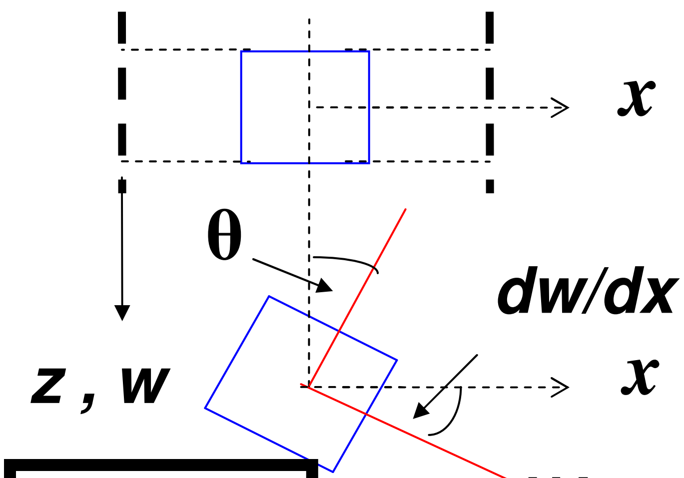
(use of penalty methods)

Elementary beam theory as constrained media problem

Principal assumption in Euler's simple beam theory (valid for thin beams):
Plane sections normal to the neutral axis before bending remain normal to it even after bending.
This means that

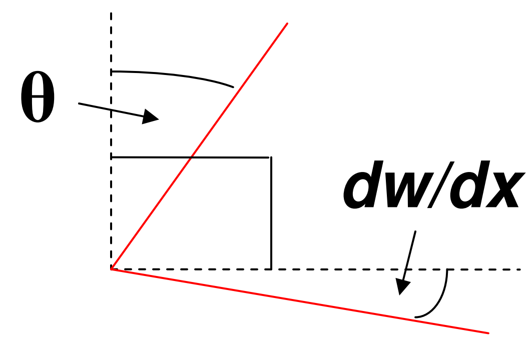
(a) bending rotation is equal to the slope of the neutral axis.
(b) shear deformation γ_{zx} is (assumed) zero.

Euler beam model



$$\boxed{\theta = dw/dx}$$

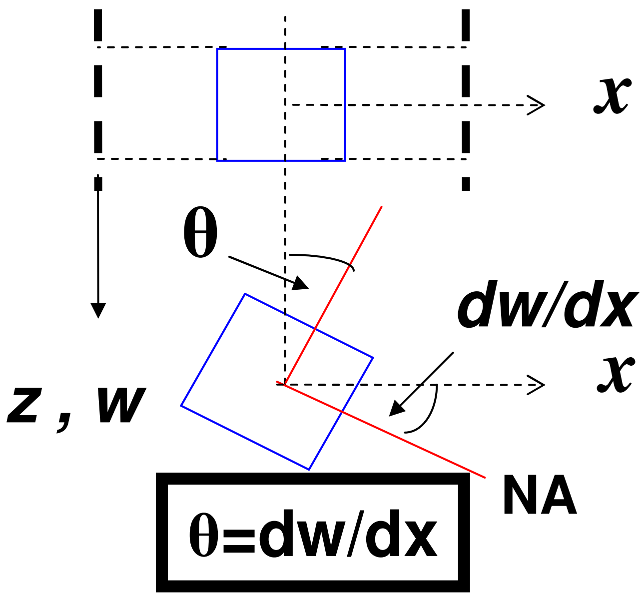
Shear flexible beam model



$$\boxed{\gamma_{zx} = \theta - dw/dx}$$

Elementary beam theory as constrained media problem (continued)

Euler beam model



The Euler beam has infinite shear rigidity κ

But the practice of using a large shear rigidity κ for thin beams creates a problem called Shear Locking in shear-flexible beam elements.

$$\delta\Pi = \delta\left[\int_0^L \frac{1}{2} EI\left(\frac{d\theta}{dx}\right)^2 dx - \int_0^L qw\,dx + \int_0^L \frac{1}{2}\kappa\left(\theta - \frac{dw}{dx}\right)^2\right] = 0$$

Equilibrium Equations

$$EI\frac{d^2\theta}{dx^2} - \kappa\left(\theta - \frac{dw}{dx}\right) = 0 \qquad ....(i)$$

$$\kappa\left(\frac{d\theta}{dx} - \frac{d^2w}{dx^2}\right) = q \qquad i.e. \qquad \frac{d}{dx}\kappa\left(\theta - \frac{dw}{dx}\right) = q \qquad ....(ii)$$

Combining (i) & (ii)

$$\frac{d}{dx}EI\frac{d^2\theta}{dx^2} - q = 0 \qquad ...(iii)$$

Boundary conditions at $x = 0$ & $x = L$

Either $EI\frac{d\theta}{dx} = 0$ *or* $\delta\theta = 0$

Either $\kappa\left(\theta - \frac{dw}{dx}\right) = 0$ *or* $\delta w = 0$

As $\kappa \to \infty \quad \theta \to \frac{dw}{dx}$

Equation(iii) reduces to $\left(EI\frac{d^4w}{dx^4} - q\right) \to 0$ 21

Lecture 2
Variational Principles in Computational Solid Mechanics

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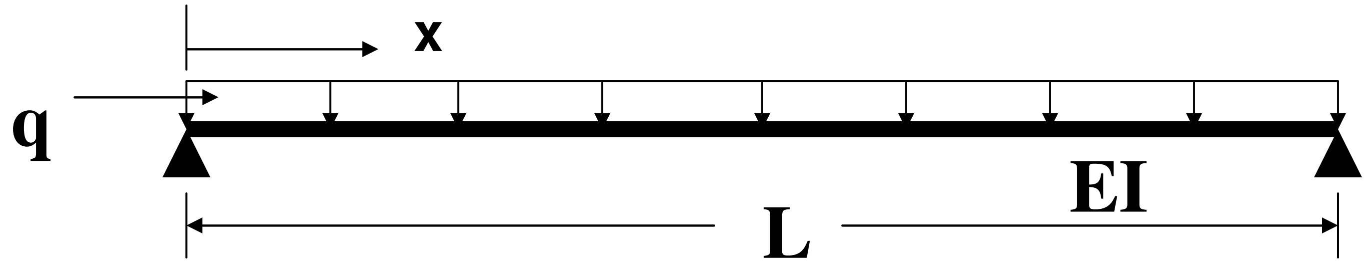
Lecture 2

Chapter 4

Element formulations using variational principles

4.1 Applications of the Rayleigh-Ritz method



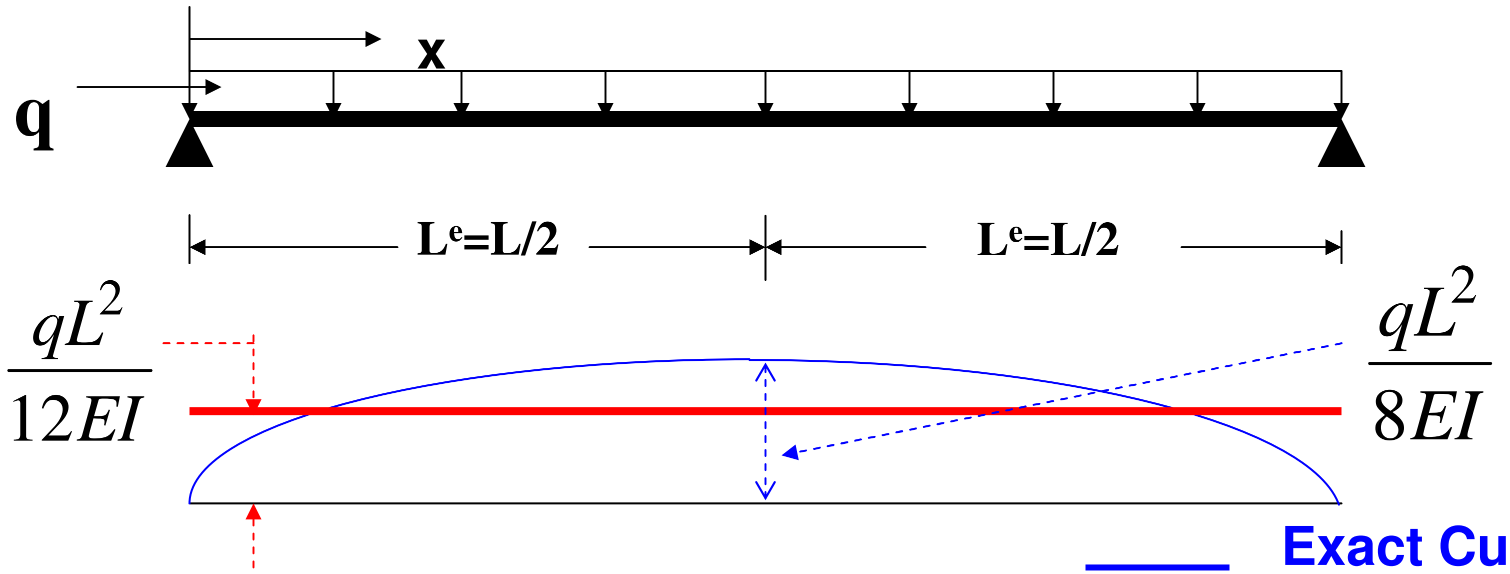
Example 1. Determine the deflection curves and bending curvature distributions of a simply supported uniform beam under u.d.l. q using the Rayleigh-Ritz method.

Use the following admissible displacement functions for approximations, satisfying the essential (kinematic) boundary conditions:

w(x=0)=0 & w(x=L)=0

$$(i) \quad w(x) = a.x(L-x)$$

$$(ii) \quad w(x) = a.\sin\frac{\pi x}{L}$$



Exact Curvature

Approximate Curvature=2a

$$(i) \quad w(x) = a.x(L-x) \quad \Rightarrow \quad \varepsilon = -\frac{d^2w}{dx^2} = 2a$$

$$\Pi = \frac{1}{2}\int_0^L EI\left(\frac{d^2w}{dx^2}\right)^2 dx - \int_0^L q.w.dx = 2a^2(EIL) - a\frac{qL^3}{6}$$

$$Equilibrium: \quad \delta\Pi = \quad 4a(\delta a).(EIL) - (\delta a).\frac{qL^3}{6} = 0$$

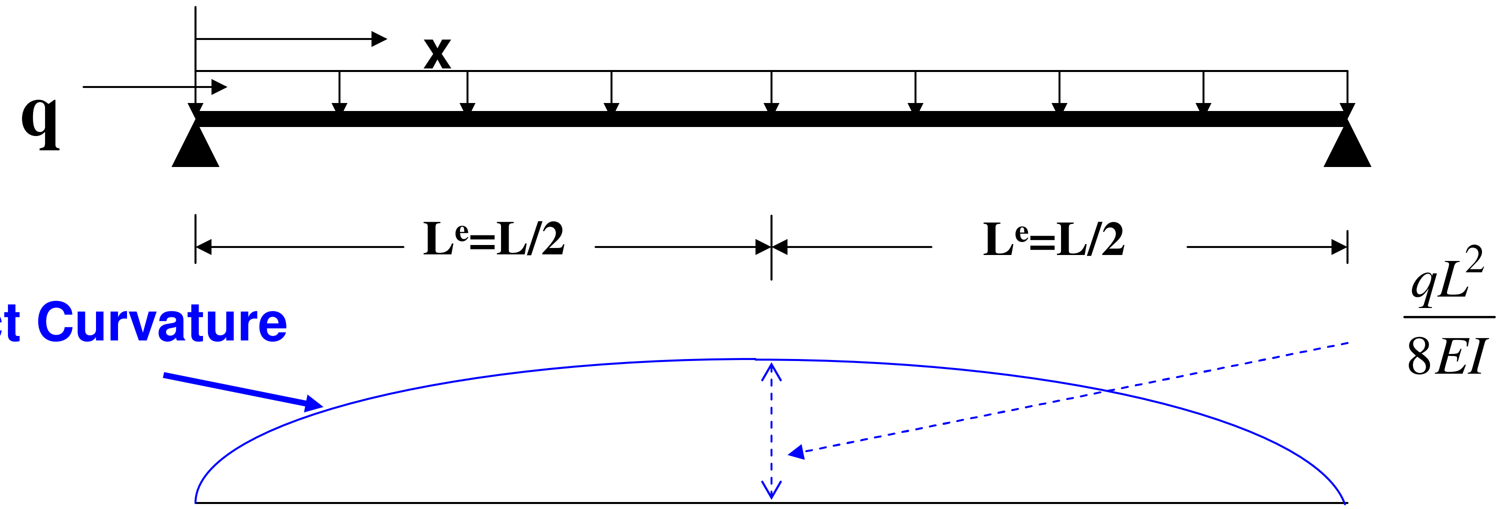
$$\Rightarrow \quad a = \frac{qL^2}{24EI} \qquad w = \frac{qL^2}{24EI}x(L-x)$$

Areas under the curvature curves are Equal. WHY?

$$A_{exact} = \frac{2L}{3}\times\frac{qL^2}{8EI} = \frac{qL^3}{12EI}$$

$$A_{approx} = L\times\frac{qL^2}{12EI} = \frac{qL^3}{12EI}$$

$$\boxed{A_{exact} = A_{approx}}$$



$$(ii) \quad w(x) = a.\sin\frac{\pi x}{L} \quad \Rightarrow \quad \varepsilon = -\frac{d^2 w}{dx^2} = a\left(\frac{\pi}{L}\right)^2 \sin\frac{\pi x}{L}$$

$$\Pi = \frac{1}{2}\int_0^L EI\left(\frac{d^2 w}{dx^2}\right)^2 dx - \int_0^L q.w.dx = a^2\left(\frac{\pi^4}{4L^4}\right)(EIL) - 2a\frac{qL}{\pi}$$

$$Equilibrium : \delta\Pi = 0 \quad \Rightarrow \quad 2a(\delta a).\left(\frac{\pi^4}{4L^4}\right)(EIL) - (2\delta a)\frac{qL}{\pi} = 0$$

$$\Rightarrow \quad a = \frac{4qL^4}{\pi^5 EI} \qquad w = \frac{4qL^4}{\pi^5 EI}\sin\frac{\pi x}{L}$$

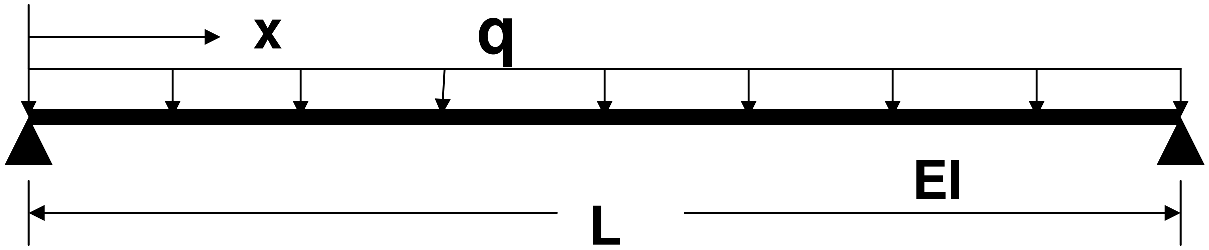
Areas under the curvature curves

$$A_{exact} = \frac{2L}{3} \times \frac{qL^2}{8EI} = \frac{qL^3}{12EI}$$

$$A_{approx} = \left(\frac{4qL^4}{\pi^5 EI}\right)\left(\frac{\pi}{L}\right)^2 \int_0^L \sin\frac{\pi x}{L} dx$$

$$A_{approx} = \frac{qL^3}{12EI}$$

$$\boxed{A_{exact} = A_{approx}}$$



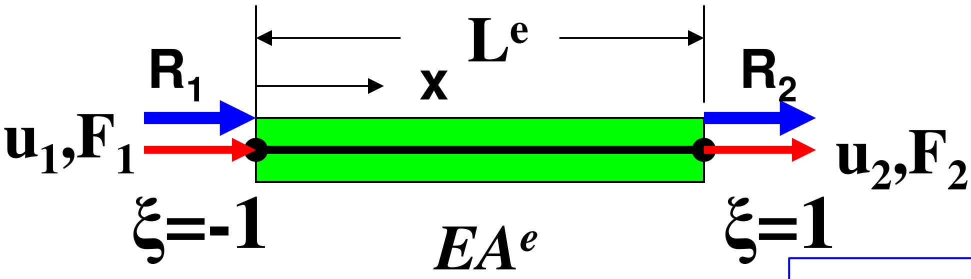
$$\|\varepsilon\|^2 = \int_0^L EI\left(-\frac{d^2w}{dx^2}\right)^2 dx$$

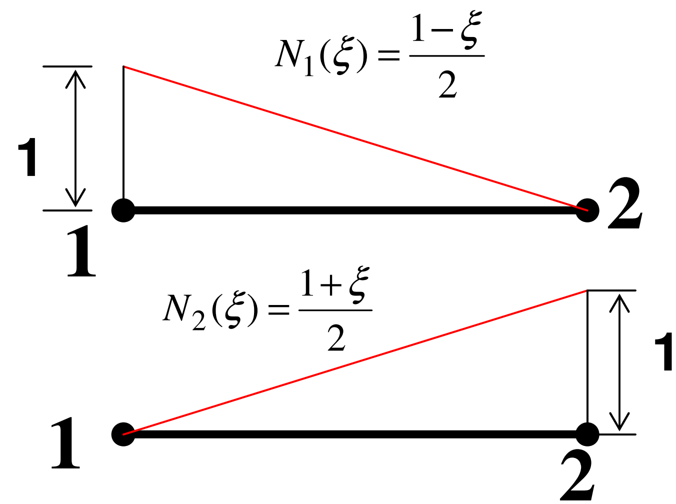
$$\|\varepsilon^h\|^2 = \int_0^L EI\left(-\frac{d^2w^h}{dx^2}\right)^2 dx$$

Displacement function and curvature (strain) (superscript *h* signifies approximated function)	**Deflection at beam center *x=L/2***	**Error of the Energy** $\|\varepsilon\|^2 - \|\varepsilon^h\|^2$	**Energy of the Error** $\|\varepsilon - \varepsilon^h\|^2$
$w^h(x) = a.x(L-x)$ $\varepsilon^h = 2a$	$(0.01041)\dfrac{qL^4}{EI}$	$\dfrac{q^2L^5}{(1440)EI}$	$\dfrac{q^2L^5}{(1440)EI}$
$w^h(x) = a.\sin(\pi x/L)$ $\varepsilon^h = a.(\pi/L)^2\sin(\pi x/L)$	$(0.01307)\dfrac{qL^4}{EI}$	$\left(\dfrac{1}{240}-\dfrac{4}{\pi^6}\right)\dfrac{q^2L^5}{EI}$	$\left(\dfrac{1}{240}-\dfrac{4}{\pi^6}\right)\dfrac{q^2L^5}{EI}$
Exact $w(x) = \left(\dfrac{q}{24EI}\right)\left\{x(L^3-2x^2L+x^3)\right\}$ $\varepsilon = \dfrac{q}{2EI}x(L-x)$	$(0.01302)\dfrac{qL^4}{EI}$	0	0

4.2 Element formulations using the principle of minimum potential energy

4.2.1 The bar element (2 nodes, 2 DOF)





Displacement function
(C⁰ continuity)

$$\boxed{\xi=\frac{2x}{L^e}-1}$$

$$\boxed{u^h=\sum_{i=1}^{2}N_i(\xi).u_i=[N_1 \quad N_2]\begin{Bmatrix}u_1\\u_2\end{Bmatrix}=[N]\{\delta^e\} \qquad N_1(\xi)=\frac{1-\xi}{2} \quad N_2(\xi)=\frac{1+\xi}{2}}$$

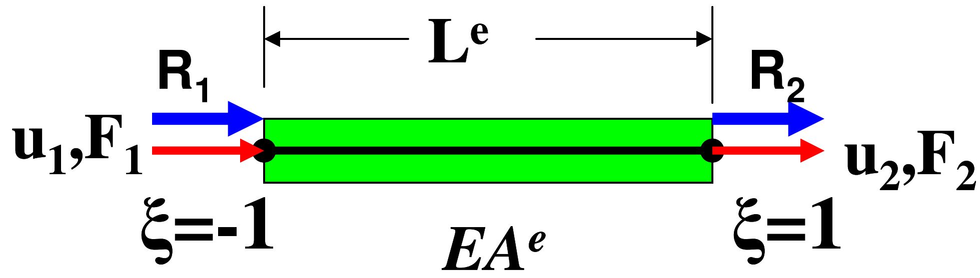
[N]: Shape function matrix **(4.1)**

Strain

$$\boxed{\varepsilon^h=\frac{u_2-u_1}{L^e}=\left[-\frac{1}{L^e} \quad \frac{1}{L^e}\right]\begin{Bmatrix}u_1\\u_2\end{Bmatrix}=[B]\{\delta^e\}} \qquad (4.2)$$

[B]: Strain-displacement matrix

The bar element



Stress

$$\sigma^h = EA^e \varepsilon^h = [D][B]\{\delta^e\} \qquad [D] = EA^e$$

Strain Energy of element

$$U^e = \frac{1}{2}\int_{x=0}^{x=L} \left\{\varepsilon^h\right\}^T \left\{\sigma^h\right\} dx = \{\delta^e\}^T \left[\frac{1}{2}\int_{\xi=-1}^{\xi=1} [B]^T [D][B]\frac{L}{2}d\xi\right]\{\delta^e\} = \frac{1}{2}\{\delta^e\}^T [K^e]\{\delta^e\}$$

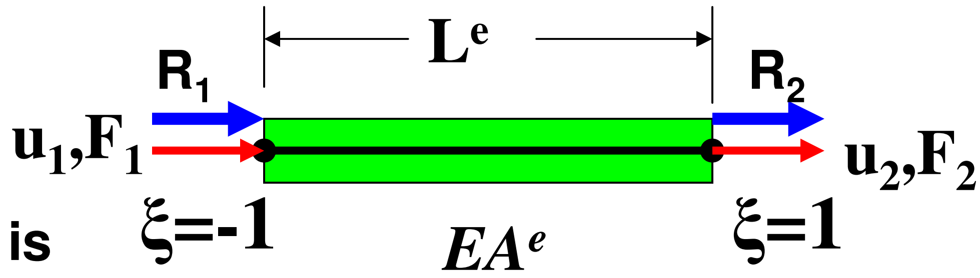
$$\textit{where} \qquad [K^e] = \int_{\xi=-1}^{\xi=1} [B]^T [D][B]\frac{L}{2}d\xi = \frac{EA^e}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \mathbf{(4.3)}$$

Work done by external loading *q(x)*

$$W^e = \int_{x=0}^{L} \{u^h\}^T q(x).dx = \{\delta^e\}^T \int_{\xi=-1}^{1} [N]^T q(\xi).\frac{L}{2}d\xi = \{\delta^e\}^T \{F^e\}$$

$$\textit{where} \qquad \{F^e\} = \int_{\xi=-1}^{1} [N]^T q(\xi).\frac{L}{2}d\xi = \begin{Bmatrix} F_1^e \\ F_2^e \end{Bmatrix}, \qquad F_i^e = \int_{\xi=-1}^{1} N_i(\xi).q(\xi).\frac{L}{2}d\xi \qquad \mathbf{(4.4)}$$

The bar element



Potential Energy of element is

$$\Pi^e = U^e - W^e$$

$$= \frac{1}{2}\{\delta^e\}^T[K^e]\{\delta^e\} - \{\delta^e\}^T\{F^e\} - \{\delta^e\}^T\{R^{h,e}\}$$

$$\{R^{h,e}\} = \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix}$$

Reaction vector

Equilibrium of element demands that $\boxed{\delta\Pi^e = 0}$

$$0 = \lfloor \delta\{\delta^e\}^T \rfloor [K^e]\{\delta^e\} - \lfloor \delta\{\delta^e\}^T \rfloor \{F^e\} - \lfloor \delta\{\delta^e\}^T \rfloor \{R^{h,e}\}$$

$$\boxed{[K^e]\{\delta^e\} = \{F^e\} + \{R^{h,e}\}} \qquad \textbf{(4.5)}$$

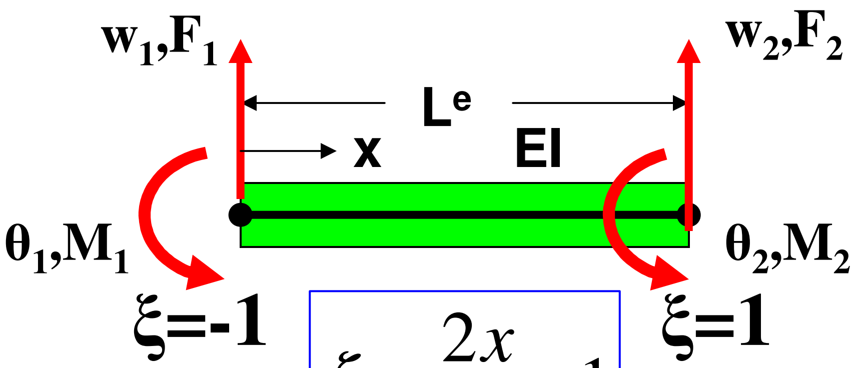
Element Stiffness

$$\boxed{[K^e] = \int_{\xi=-1}^{\xi=1} [B]^T[D][B]\frac{L}{2}d\xi = \frac{EA^e}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}$$

Generalized force vector

$$\boxed{F_i^e = \int_{\xi=-1}^{1} N_i(\xi).q(\xi).\frac{L}{2}d\xi}$$

4.2.2 The Euler beam element (2 nodes, 4 DOF)



$$\boxed{\xi = \frac{2x}{L^e} - 1}$$

Displacement function (C¹ continuity)

$$w^h = \sum_{i=1}^{4} N_i(\xi).\delta_i = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = [N]\{\delta^e\} \qquad \textbf{(4.6)}$$

$$N_1 = \frac{1}{4}\left\{2 + \xi(\xi^2 - 3)\right\} \quad N_2 = \frac{L^e}{8}(\xi+1)(\xi-1)^2 \quad N_3 = \frac{1}{4}\left\{2 - \xi(\xi^2 - 3)\right\} \quad N_2 = \frac{L^e}{8}(\xi+1)^2(\xi-1)$$

[N]: Shape function matrix

The Euler beam element

Strain and stress resultant vectors **[B]: Strain-displacement matrix**

$$\varepsilon^h = \frac{d^2 w^h}{dx^2} = \frac{4}{\left(L^e\right)^2}\frac{d^2 w^h}{d\xi^2} = \frac{1}{\left(L^e\right)^2}\left[6\xi \quad L^e(3\xi-1) \quad -6\xi \quad L^e(3\xi+1)\right]\{\delta^e\} = [B]\{\delta^e\}$$

$$\sigma^h = M^h = EI^e \varepsilon^h = [D][B]\{\delta^e\} \qquad [D] = EI^e \tag{4.7}$$

Strain Energy of element

$$U^e = \frac{1}{2}\int_{x=0}^{x=L}\left\{\varepsilon^h\right\}^T\left\{\sigma^h\right\}dx = \{\delta^e\}^T\left[\frac{1}{2}\int_{\xi=-1}^{\xi=1}[B]^T[D][B]\frac{L}{2}d\xi\right]\{\delta^e\} = \frac{1}{2}\{\delta^e\}^T[K^e]\{\delta^e\}$$

$$where \qquad [K^e] = \int_{\xi=-1}^{\xi=1}[B]^T[D][B]\frac{L}{2}d\xi$$

Stiffness matrix

$$\left[K^e\right] = \begin{bmatrix} 12(EI^e/L^{e^3}) & 6(EI^e/L^{e^2}) & -12(EI^e/L^{e^3}) & 6(EI^e/L^{e^2}) \\ 6(EI^e/L^{e^2}) & 4(EI^e/L^e) & -6(EI^e/L^{e^2}) & 2(EI^e/L^e) \\ -12(EI^e/L^{e^3}) & -6(EI^e/L^{e^2}) & 12(EI^e/L^{e^3}) & -6(EI^e/L^{e^2}) \\ 6(EI^e/L^{e^2}) & 2(EI^e/L^e) & -6(EI^e/L^{e^2}) & 4(EI^e/L^e) \end{bmatrix} \tag{4.8}$$

The Euler beam element

Work done by external loading *q(x)*

$$W^e = \int_{x=0}^{L} \{w^h\}^T q(x).dx = \{\delta^e\}^T \int_{\xi=-1}^{1} [N]^T q(\xi).\frac{L}{2} d\xi = \{\delta^e\}^T \{F^e\}$$

$$where \qquad \{F^e\} = \int_{\xi=-1}^{1} [N]^T q(\xi).\frac{L}{2} d\xi = \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} \qquad \textbf{(4.9)}$$

Potential Energy of element is

$$\Pi^e = U^e - W^e$$

$$= \frac{1}{2}\{\delta^e\}^T [K^e]\{\delta^e\} - \{\delta^e\}^T \{F^e\} - \{\delta^e\}^T \{R^{h,e}\}$$

$$\{R^{h,e}\} = \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix}$$

Equilibrium of element demands that $\boxed{\delta\Pi^e = 0}$ **Reaction vector**

$$\boxed{[K^e]\{\delta^e\} = \{F^e\} + \{R^{h,e}\}} \qquad \textbf{(4.10)}$$

4.3 The general methods of element formulation

The Rayleigh Ritz Method | **The Galerkin Method**

Steps

1. Use a displacement interpolation

$$\{u^h\} = [N]\{\delta^e\}$$

2. Develop strain and stress resultants.

$$\varepsilon^h = [B]\{\delta^e\}$$
$$\sigma^h = [D]\varepsilon^h = [D][B]\{\delta^e\}$$

Differential eqn.

$$\mathcal{L}(u) = q$$

3. Generate potential function

$$\Pi^e = U^e - W^e$$
$$= \frac{1}{2}\{\delta^e\}^T[K^e]\{\delta^e\} - \{\delta^e\}^T\{F^e\} - \{\delta^e\}^T\{R^{h,e}\}$$

Virtual work

$$\int_\Omega [N]^T . \left\{\mathcal{L}\left(\{u^h\}\right) - q\right\} d\Omega = 0 \quad \textbf{(4.12)}$$

4. Use PMPE

$$\delta\Pi^e = 0$$

$$[K^e]\{\delta^e\} = \{F^e\} + \{R^{h,e}\}$$

$$[K^e] = \int_e [B]^T[D][B]dx = \int_e [B]^T[D][B]\det[J]d\xi$$

$$\{F^e\} = \int_e [N]^T q(x).dx = \int_e [N]^T q(\xi).\det[J]d\xi \quad \textbf{(4.11)}$$

4.4 Some specimens from the Lagrangian family of C^0 elements

Displacement approximation in an element through Lagrangian interpolation

$$u^h = N_1(x)u_1 + N_2(x)u_2 + ...N_n(x)u_n = \sum_{i=1}^{n} N_i(x).u_i = [N]\{\delta^e\}$$

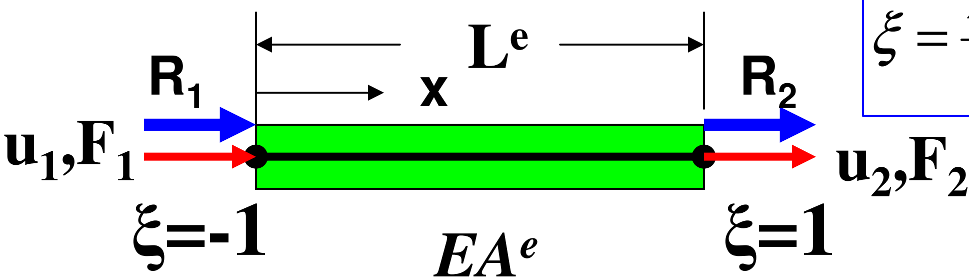
$$N_1(x) = \frac{(x_2 - x)(x_3 - x)...(x_n - x)}{(x_2 - x_1)(x_3 - x_1)...(x_n - x_1)} = \frac{\prod_{j=2}^{n}(x_j - x)}{\prod_{j=2}^{n}(x_j - x_1)} \qquad N_i(x) = \frac{\prod_{j\neq i}^{n}(x_j - x)}{\prod_{j\neq i}^{n}(x_j - x_i)}$$

$$\sum_{i=1}^{n} N_i(\xi) = 1 \qquad \textbf{(4.13)}$$

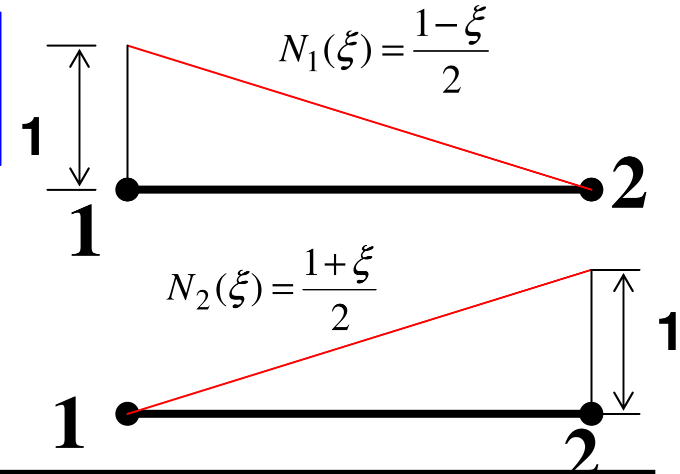
Such elements are of C^0 continuity because across the element boundary, only the function interpolated is continuous, but its derivatives are not continuous.

$$\sum_{i=1}^{n} N_i(\xi) = 1$$ **Ensures rigid body displacement of element**

(a) For the two noded bar element, a linear Lagrangian is used for displacement, hence a constant strain over the element results:



$$\xi = \frac{2x}{L^e} - 1$$



Displacement function

$$u^h = \sum_{i=1}^{2} N_i(\xi).u_i = [N_1 \quad N_2]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = [N]\{\delta^e\} \qquad N_1(\xi) = \frac{1-\xi}{2} \qquad N_2(\xi) = \frac{1+\xi}{2}$$

$$\sum_{i=1}^{2} N_i(\xi) = 1 \qquad (4.14)$$

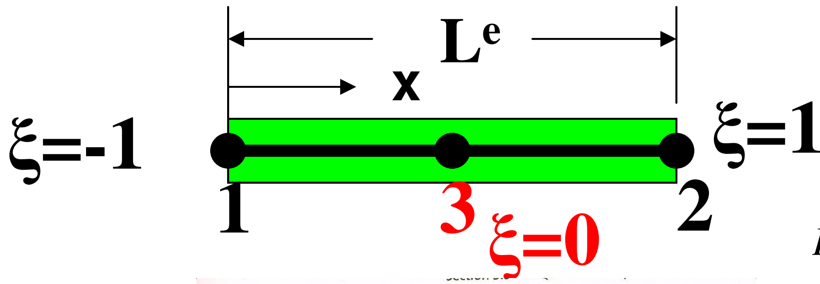
[N]: Shape function matrix

Strain

$$\varepsilon^h = \frac{u_2 - u_1}{L^e} = \begin{bmatrix} -\frac{1}{L^e} & \frac{1}{L^e} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = [B]\{\delta^e\} \qquad (4.15)$$

[B]: Strain-displacement matrix

(b) For the three noded bar element, a quadratic Lagrangian is used for displacement, hence a linear strain over the element results:



$$\xi = \frac{2x}{L^e} - 1$$

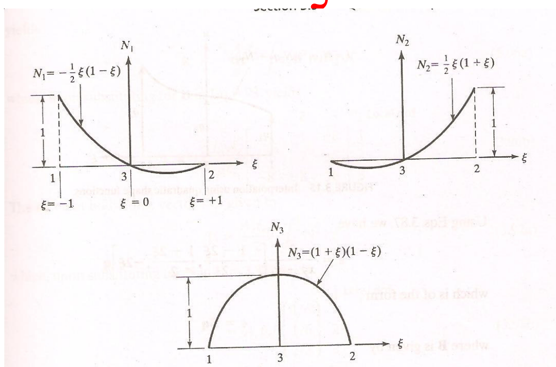


FIGURE 3.12 Shape functions N_1, N_2, and N_3.

Element Stiffness Matrix

$$[K^e] = \frac{1}{2}\int_{\xi=-1}^{1} [B]^T EA^e [B]\left(\frac{dx}{d\xi}\right)d\xi = \frac{EA^e}{3L^e}\begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix}$$

Element force vector

$$\{F^e{}_{Applied}\} = \int_{-1}^{1}[N]^T q.\left(\frac{dx}{d\xi}\right)d\xi = qL^e\begin{Bmatrix} 1/6 \\ 1/6 \\ 2/3 \end{Bmatrix} \qquad \textbf{(4.17)}$$

q=force p.u.length

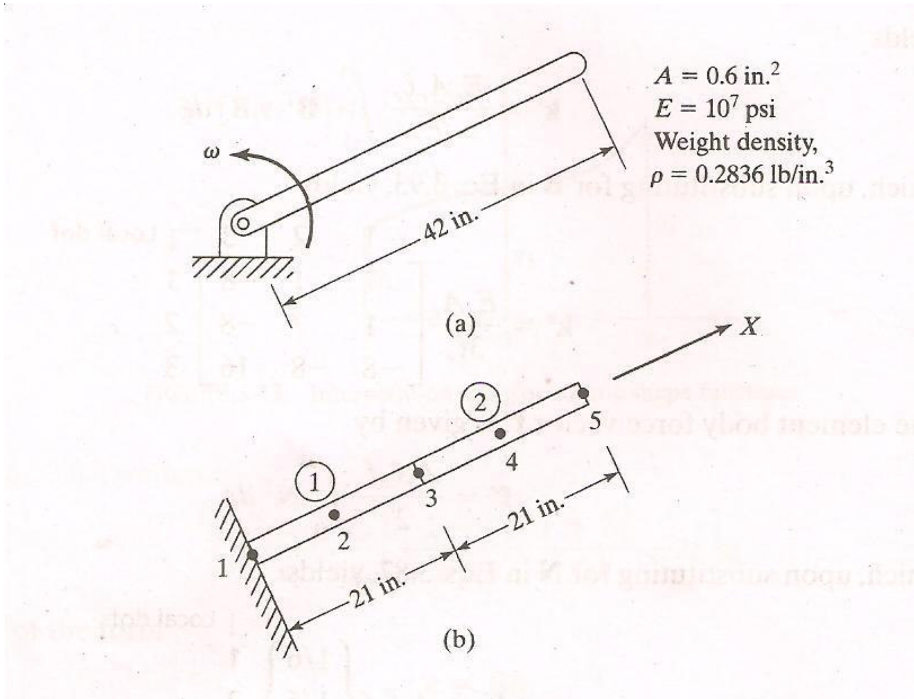
$$u^h = \sum_{i=1}^{3} N_i(\xi).u_i = [N_1 \quad N_2 \quad N_3]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = [N]\{\delta^e\} \qquad N_1 = -\frac{1}{2}\xi(1-\xi), \quad N_2 = \frac{1}{2}\xi(1+\xi), \quad N_3 = (1-\xi^2)$$

$$\textbf{(4.16)}$$

$$\sum_{i=1}^{3} N_i(\xi) = 1$$

$$\varepsilon^h = \frac{du^h}{dx} = \frac{du^h}{d\xi}\frac{d\xi}{dx} = \frac{2}{L^e}\frac{du^h}{d\xi} = \frac{2}{L^e}\left[\frac{dN_1}{d\xi} \quad \frac{dN_2}{d\xi} \quad \frac{dN_3}{d\xi}\right]\{\delta^e\} = \left[\left(\frac{2}{L^e}\right)\left(-\frac{1-2\xi}{2}\right), \quad \left(\frac{2}{L^e}\right)\left(\frac{1+2\xi}{2}\right), \quad \left(\frac{2}{L^e}\right)(-2\xi)\right]\{\delta^e\}$$

$$\varepsilon^h = [B]\{\delta^e\}$$



Example 2. **A robotic arm is rotating at a constant angular velocity ω=30 rad/sec. Determine the axial stress distribution from centrifugal forces (using two quadratic elements).**

Assembly of stiffness matrices

$$[K_G] = \sum_{e=1}^{2}[K^e] = \frac{10^7 \times 0.6}{3\times 21}\begin{array}{c}\begin{bmatrix} 7 & -8 & 1 & 0 & 0 \\ -8 & 16 & -8 & 0 & 0 \\ 1 & -8 & 14 & -8 & 1 \\ 0 & 0 & -8 & 16 & -8 \\ 0 & 0 & 1 & -8 & 7 \end{bmatrix}\end{array}\begin{array}{c}(1)\\(2)\\(3)\\(4)\\(5)\end{array}$$

Element stiffness matrices

$$[K^1] = \frac{10^7 \times 0.6}{3\times 21}\begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix}\begin{array}{c}(1)\\(3)\\(2)\end{array}$$

$$[K^2] = \frac{10^7 \times 0.6}{3\times 21}\begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix}\begin{array}{c}(3)\\(5)\\(4)\end{array}$$

Element force vectors

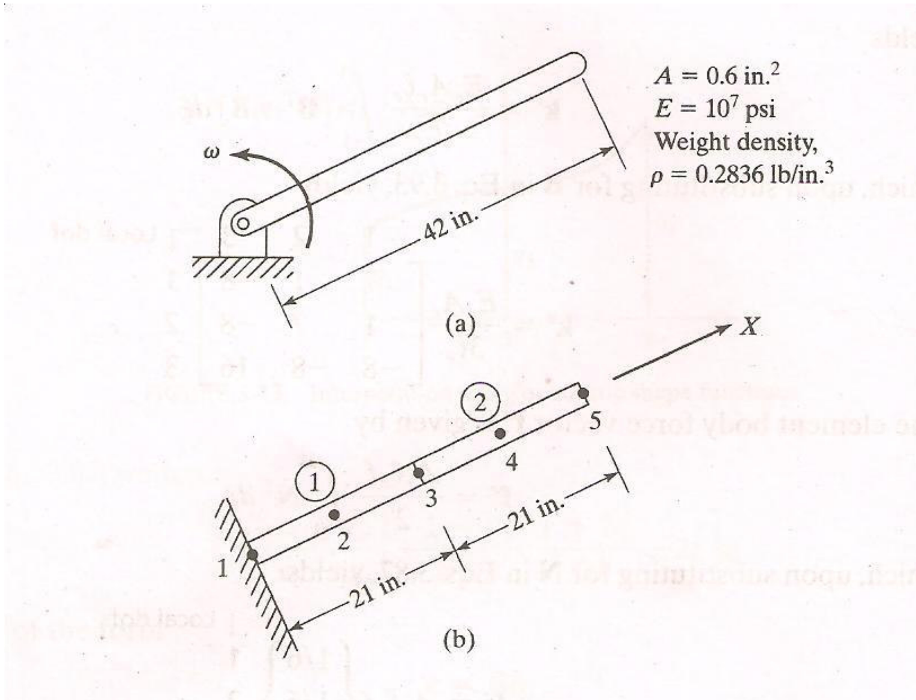
$$\{F^e{}_{applied}\} = \int_{-1}^{1}[N]^T q.\left(\frac{dx}{d\xi}\right)d\xi$$

$$\frac{dx}{d\xi} = \frac{L^e}{2}$$

$$\boxed{\xi = \frac{2x}{L^e} - 1}$$

Computation of element force vectors with averaged body force

$$\boxed{f^{e}{}_{Average} = \frac{\rho(x^{e}{}_{mean})\omega^2}{g}} \quad \textbf{lb/in}^3$$



$$\left(f^{(1)}{}_{Average}\right) = \frac{0.2836 \times (10.5) \times 30^2}{32.2 \times 12} = 6.94$$

$$\{F^{1}{}_{applied}\} = \left(f^{(1)}{}_{Average}\right) A^e L^e \begin{Bmatrix} 1/6 \\ 1/6 \\ 2/3 \end{Bmatrix} \begin{matrix} (1) \\ (2) \\ (3) \end{matrix}$$

$$\left(f^{(2)}{}_{Average}\right) = \frac{0.2836 \times (31.5) \times 30^2}{32.2 \times 12} = 20.81$$

$$\{F^{2}{}_{applied}\} = \left(f^{(2)}{}_{Average}\right) A^e L^e \begin{Bmatrix} 1/6 \\ 1/6 \\ 2/3 \end{Bmatrix} \begin{matrix} (3) \\ (5) \\ (4) \end{matrix}$$

The global (applied) force vector after force assembly

$$\boxed{\{F^{G}{}_{Applied}\} = \begin{bmatrix} 14.57 & 58.26 & 58.26 & 174.79 & 43.70 \end{bmatrix}^T}$$

Solving for displacements ($u_1=0$). Eliminating u_1

$$\frac{10^7\times0.6}{63}\begin{bmatrix}16 & -8 & 0 & 0\\ -8 & 14 & -8 & 1\\ 0 & -8 & 16 & -8\\ 0 & 1 & -8 & 7\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\\ u_4\\ u_5\end{Bmatrix}=\begin{Bmatrix}58.26\\ 58.26\\ 174.79\\ 43.7\end{Bmatrix}$$

$$\{\delta^G\}=10^{-3}\times\left[0 \quad 0.5735 \quad 1.0706 \quad 1.4147 \quad 1.5294\right]^T$$

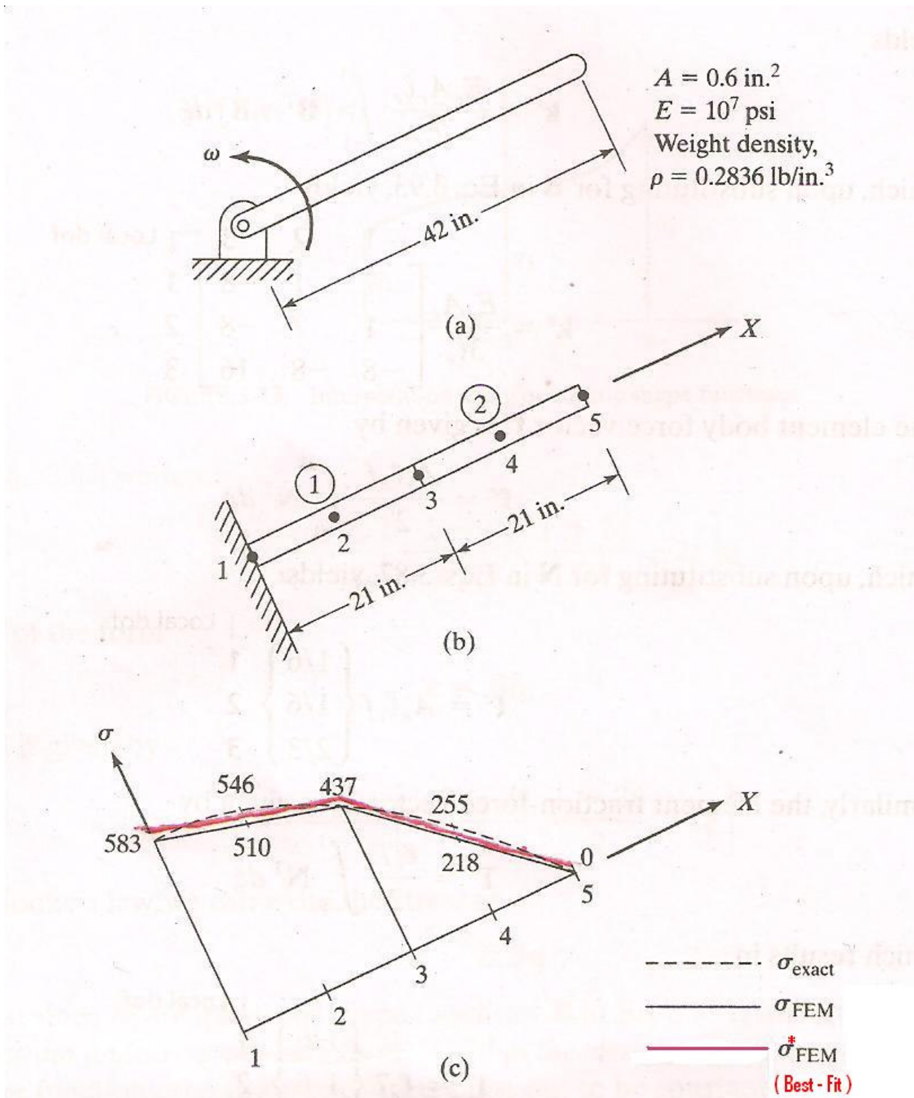


FIGURE E3.7

Element strains and stresses

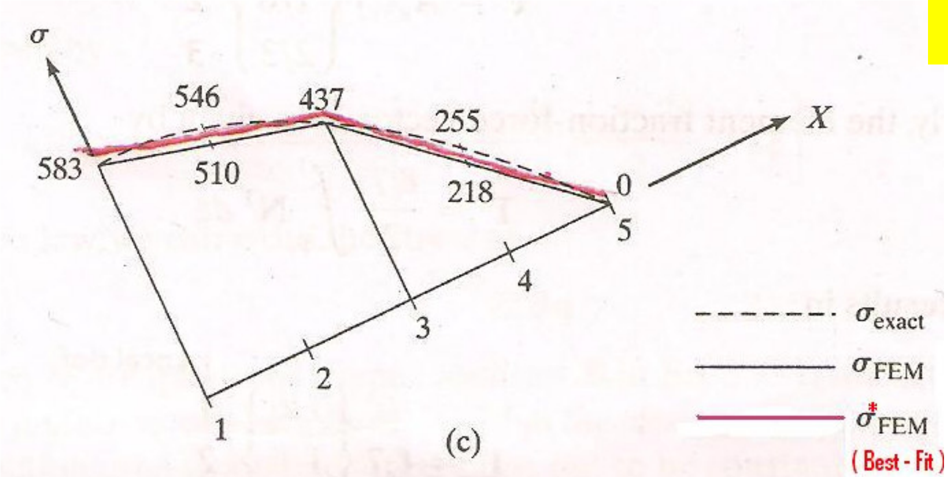
$$\varepsilon^h=[B]\{\delta^e\}=\left[\left(\frac{2}{L^e}\right)\left(-\frac{1-2\xi}{2}\right),\ \left(\frac{2}{L^e}\right)\left(\frac{1+2\xi}{2}\right),\ \left(\frac{2}{L^e}\right)(-2\xi)\right]\{\delta^e\}$$

$$\sigma^h=E\varepsilon^h=10^7\times\left[\left(\frac{2}{L^e}\right)\left(-\frac{1-2\xi}{2}\right),\ \left(\frac{2}{L^e}\right)\left(\frac{1+2\xi}{2}\right),\ \left(\frac{2}{L^e}\right)(-2\xi)\right]\{\delta^e$$

$$Exact \qquad \sigma=\frac{\rho\omega^2L^2}{2g}\left\{1-\left(\frac{x}{L}\right)^2\right\}$$

Question

Why does the stress obtained from the present FE solution does not coincide with the best-fit ?



Question

Why does the stress obtained from the present solution does not coincide with the best-fit ?

From the differential equation

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right)=q(x)$$

Exact stress (quadratic) is

$$\sigma = E\frac{du}{dx} = -\frac{1}{A^e}\int_e (q)dx = -\frac{1}{A^e}\int_e (fA^e)dx$$

$$\boxed{Exact \qquad \sigma = \frac{\rho\omega^2 L^2}{2g}\left\{1-\left(\frac{x}{L}\right)^2\right\}}$$

With the consistent force vector, we can get the FE stress as the linear best-fit to the analytical stress in each element.

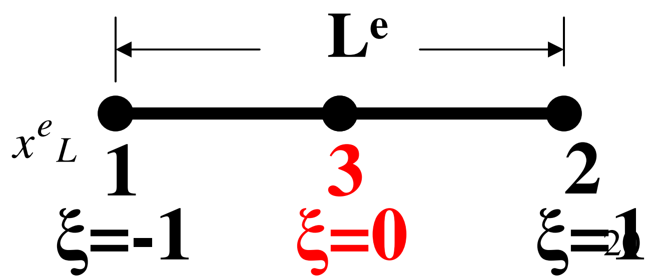
This implies that the true body force f should be linearly varying (and not constant average) within the element

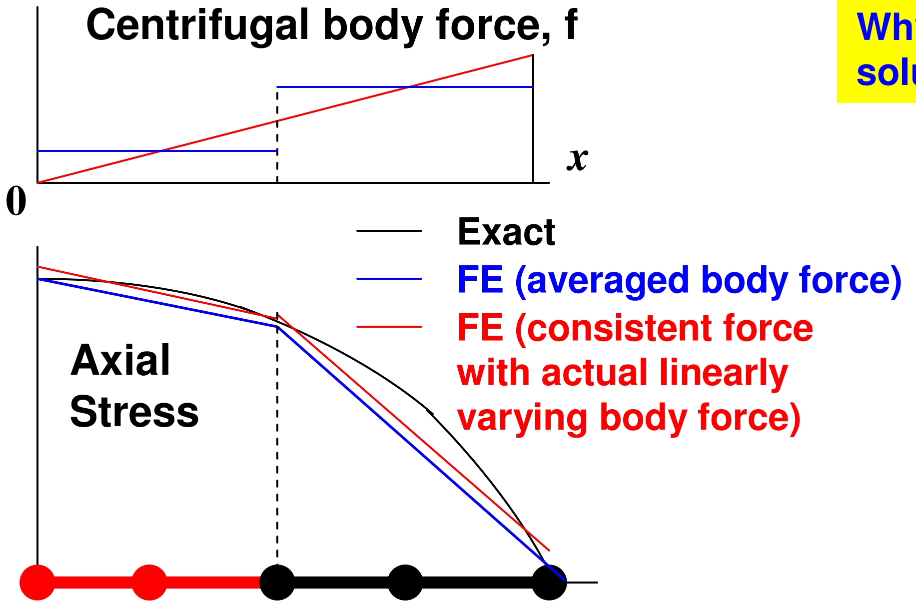
$$f = \frac{\rho\omega^2 x}{g} \qquad q = fA^e = \frac{\rho\omega^2 x}{g}A^e$$

This is the consistent force vector for each element.

$$\{F^e{}_{applied}\} = \int_0^{L^e}[N]^T q(x)dx = \int_{-1}^{1}[N]^T q(\xi).\left(\frac{L^e}{2}\right)d\xi$$

$$q(\xi) = \frac{\rho\omega^2 x(\xi)}{g}A^e = \frac{\rho\omega^2[x^e{}_L + L^e(\xi+1)/2]}{g}A^e$$





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Why does the stress obtained from the present solution does not coincide with the best-fit ?

From the differential equation

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right)=q(x)$$

Exact stress (quadratic) is

$$\sigma = E\frac{du}{dx} = -\frac{1}{A^e}\int_e (q)dx = -\frac{1}{A^e}\int_e (fA^e)dx$$

$$Exact \qquad \sigma = \frac{\rho\omega^2 L^2}{2g}\left\{1-\left(\frac{x}{L}\right)^2\right\}$$

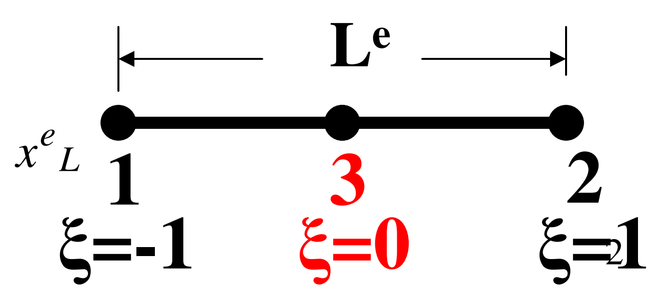
This implies that the true body force f should be linearly varying (and not constant average) within the element

$$f = \frac{\rho\omega^2 x}{g} \qquad q = fA^e = \frac{\rho\omega^2 x}{g}A^e$$

This is the consistent force vector for each element.

$$\{F^e{}_{applied}\} = \int_0^{L^e}[N]^T q(x)dx = \int_{-1}^{1}[N]^T q(\xi).\left(\frac{L^e}{2}\right)d\xi$$

$$q(\xi) = \frac{\rho\omega^2 x(\xi)}{g}A^e = \frac{\rho\omega^2[x^e{}_L + L^e(\xi+1)/2]}{g}A^e$$



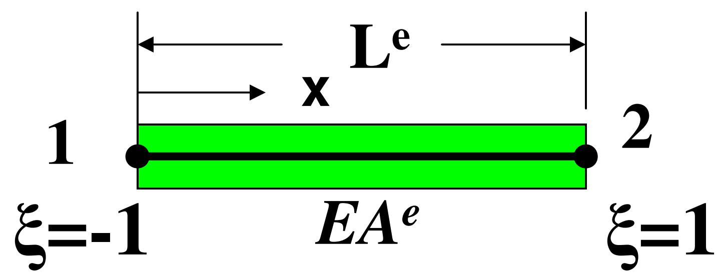
4.5 Some specimens from the Isoparametric family of elements

Displacement approximation and Geometry of element described by the same set of Lagrangian interpolation functions

$$\{u^h\} = [N]\{\delta^e\}$$
$$\{x\} = [N]\{x^e\} \qquad \textbf{(4.18)}$$

Advantages:
Curved/inclined outlines/surfaces of elements can be suitably modeled.

(a) The aforementioned two-noded bar element is a linear isoparametric element.



$$u^h = \sum_{i=1}^{2} N_i(\xi).u_i = [N_1 \quad N_2]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = [N]\{\delta^e\} \qquad N_1(\xi) = \frac{1-\xi}{2} \quad N_2(\xi) = \frac{1+\xi}{2}$$

$$x = \sum_{i=1}^{2} N_i(\xi).x_i = [N_1 \quad N_2]\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = [N]\{x^e\}$$

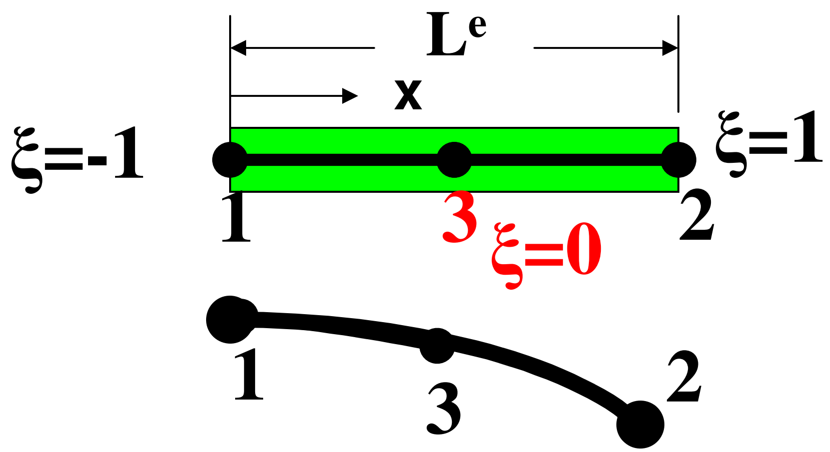
(b) The three-noded (quadratic) bar element with quadratic geometry description

$$u^h = \sum_{i=1}^{3} N_i(\xi).u_i = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = [N]\{\delta^e\}$$

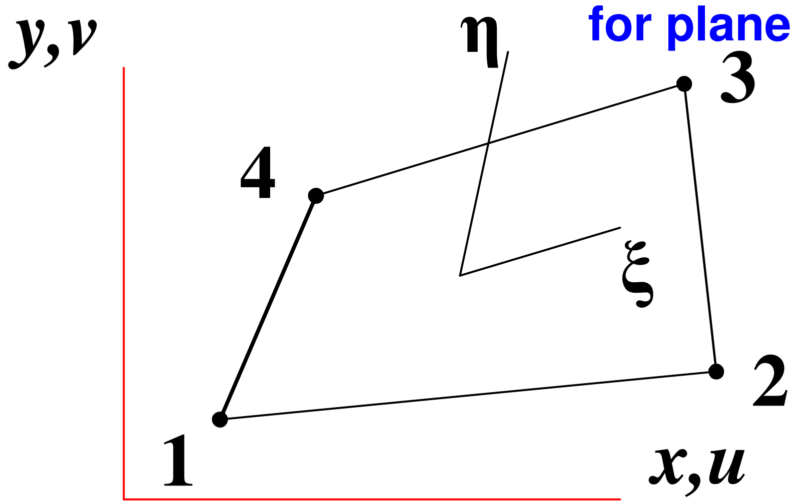
$$N_1 = -\frac{1}{2}\xi(1-\xi), \quad N_2 = \frac{1}{2}\xi(1+\xi), \quad N_3 = (1-\xi^2)$$

$$x = \sum_{i=1}^{3} N_i(\xi).x_i = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = [N]\{x^e\}$$

$$\varepsilon^h = \frac{du^h}{dx} = \frac{d\xi}{dx}\frac{du^h}{d\xi} = \frac{(du^h / d\xi)}{dx / d\xi} = [B]\{\delta^e\} \qquad \textbf{(4.19)}$$



(c) The four-noded (linear) quadrilataral element for plane stress problem



Node,i	ξ_i	η_i
1	-1	-1
2	+1	-1
3	+1	+1
4	-1	+1

Shape functions

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta), \qquad N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

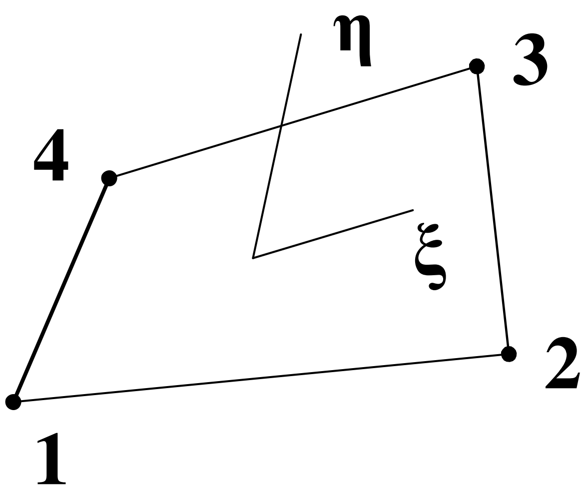
$$N_3 = \frac{1}{4}(1+\xi)(1+\eta), \qquad N_2 = \frac{1}{4}(1-\xi)(1+\eta)$$

Displacement

$$\begin{Bmatrix} u^h \\ v^h \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = [N]\{\delta^e\} \qquad (4.20)$$

Geometry

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} \sum_{i=1}^{4} N_i x_i \\ \sum_{i=1}^{4} N_i y_i \end{bmatrix} = [N]\{X^e\} \qquad (4.21)$$



(c) The four-noded (linear) quadrilatаral element for plane stress problem

Element stress

$$\{\sigma^h\}=\begin{Bmatrix}\sigma_x{}^h\\ \sigma_y{}^h\\ \tau_{xy}{}^h\end{Bmatrix}=\frac{E}{(1-\mu^2)}\begin{bmatrix}1 & \mu & 0\\ \mu & 1 & 0\\ 0 & 0 & (1-\mu)/2\end{bmatrix}\begin{Bmatrix}\varepsilon_x{}^h\\ \varepsilon_y{}^h\\ \gamma_{xy}{}^h\end{Bmatrix}=[D][B]\{\delta^e\}$$

Element strain

$$\{\varepsilon^h\}=\begin{Bmatrix}\varepsilon_x{}^h\\ \varepsilon_y{}^h\\ \gamma_{xy}{}^h\end{Bmatrix}=\begin{Bmatrix}\dfrac{\partial u^h}{\partial x}\\ \dfrac{\partial v^h}{\partial y}\\ \dfrac{\partial u^h}{\partial y}+\dfrac{\partial v^h}{\partial x}\end{Bmatrix}=\begin{bmatrix}\partial N_1/\partial x & 0 & \partial N_2/\partial x & 0 & \partial N_3/\partial x & 0 & \partial N_4/\partial x & 0\\ 0 & \partial N_1/\partial y & 0 & \partial N_2/\partial y & 0 & \partial N_3/\partial y & 0 & \partial N_4/\partial y\\ \partial N_1/\partial y & \partial N_1/\partial x & \partial N_2/\partial y & \partial N_2/\partial x & \partial N_3/\partial y & \partial N_3/\partial x & \partial N_4/\partial y & \partial N_4/\partial x\end{bmatrix}\begin{Bmatrix}u_1\\ v_1\\ u_2\\ v_2\\ u_3\\ v_3\\ u_4\\ v_4\end{Bmatrix}=[B]\{\delta^e\}$$

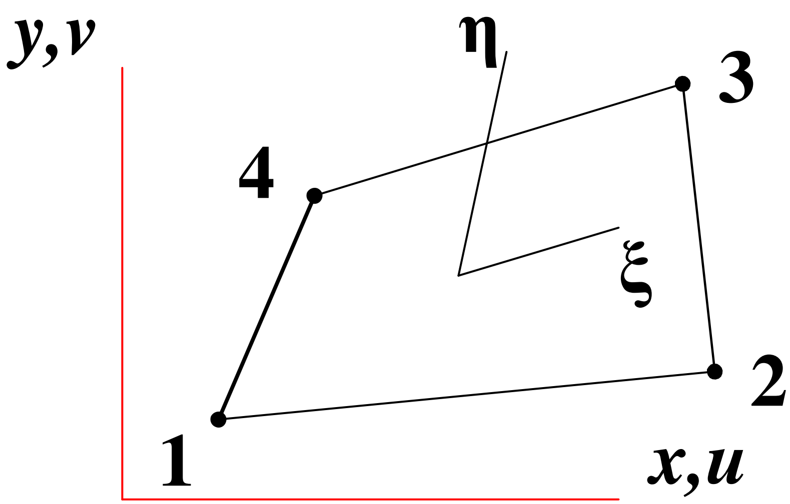
(4.22)

Jacobian

$$[J]=\frac{\partial(x,y)}{\partial(\xi,\eta)}=\begin{bmatrix}\partial x/\partial\xi & \partial y/\partial\xi\\ \partial x/\partial\eta & \partial y/\partial\eta\end{bmatrix}=\begin{bmatrix}\sum_{i=1}^{4}(\partial N_i/\partial\xi)x_i & \sum_{i=1}^{4}(\partial N_i/\partial\xi)y_i\\ \sum_{i=1}^{4}(\partial N_i/\partial\eta)x_i & \sum_{i=1}^{4}(\partial N_i/\partial\eta)y_i\end{bmatrix}$$

(4.23)

(c) The four-noded (linear) quadrilataral element for plane stress problem



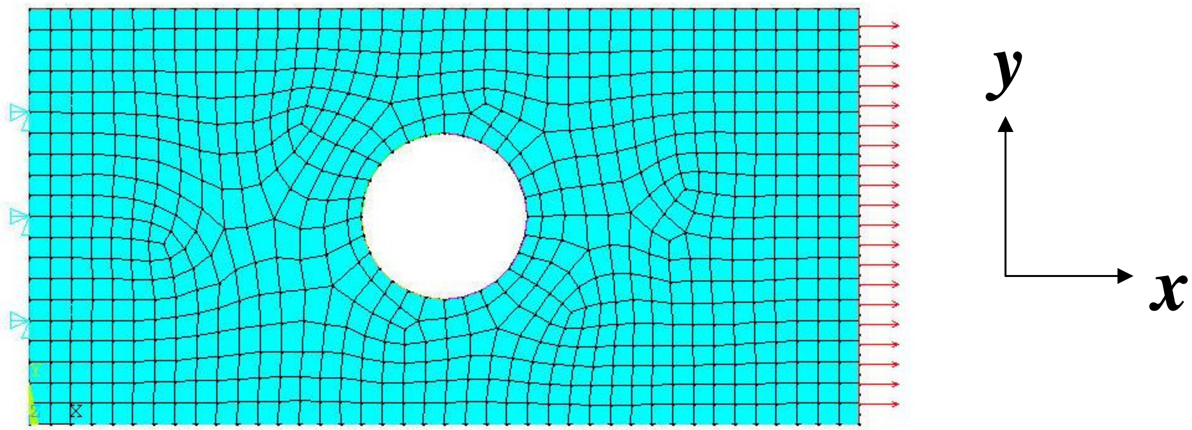
$$\begin{Bmatrix} \partial N_i / \partial x \\ \partial N_i / \partial y \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \partial N_i / \partial \xi \\ \partial N_i / \partial \eta \end{Bmatrix}$$

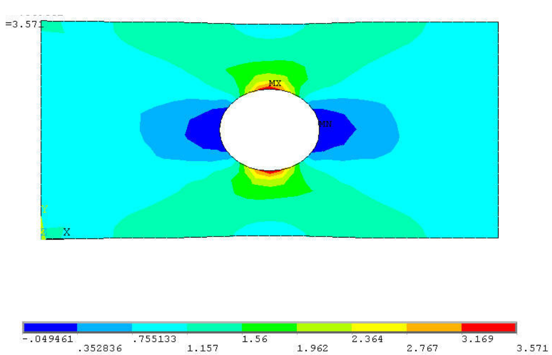
Element stiffness matrix:

$$\boxed{[K^e] = \iint_e [B]^T [D][B].t.dxdy = \int_{-1}^{1}\int_{-1}^{1} [B]^T [D][B].t \det[J].d\xi d\eta} \quad \textbf{(4.24)}$$

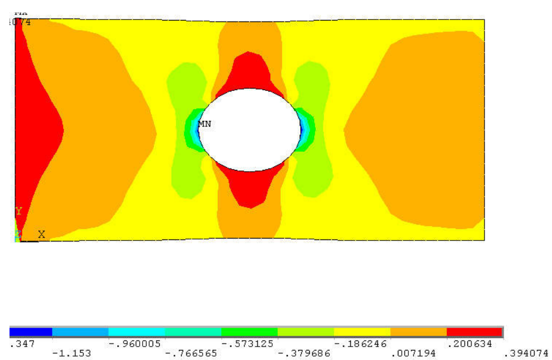
t=element thickness

Example 3. Plane stress analysis of a plate with a hole using quad4 elements

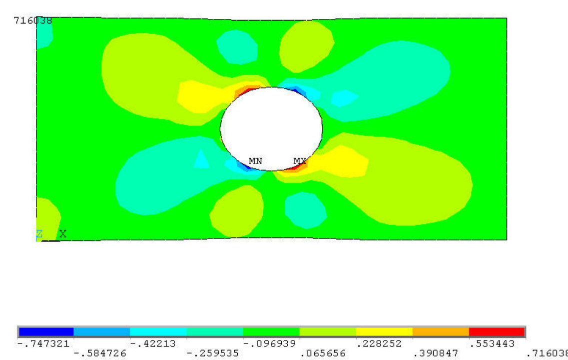




σ_x



σ_y



τ_{xy}

4.6 Numerical Integration by Gauss Quadrature

If f(ξ) is a polynomial of degree (2n-1) or less, then it is exactly integrated within the domain $-1 \le \xi \le 1$ by the n-point Gaussian quadrature rule.

$$f(\xi) = p_{2n}(\xi) = a_1 + a_2\xi + a_3\xi^3 ..... + a_{2n}\xi^{2n-1} \qquad -1 \le \xi \le 1$$

$$\int_{-1}^{1} f(\xi)d\xi = \sum_{i=1}^{n} w_i f(\xi_i) \qquad \textbf{(4.25)}$$

$$\int_{-1}^{1} f(\xi)\, d\xi \approx \sum_{i=1}^{n} w_i f(\xi_i)$$

Number of points, n	Location, ξ_i	Weights, w_i
1	0.0	2.0
2	$\pm 1/\sqrt{3} = \pm 0.5773502692$	1.0
3	$\pm$0.7745966692	0.5555555556
	0.0	0.8888888889
4	$\pm$0.8611363116	0.3478548451
	$\pm$0.3399810436	0.6521451549
5	$\pm$0.9061798459	0.2369268851
	$\pm$0.5384693101	0.4786286705
	0.0	0.5688888889
6	$\pm$0.9324695142	0.1713244924
	$\pm$0.6612093865	0.3607615730
	$\pm$0.2386191861	0.4679139346